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Distinguishing chromatic numbers of complements of Cartesian products of complete graphs

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ABSTRACT

The distinguishing chromatic number of a graph, G, is the minimum number of colours required to properly colour the vertices of G so that the only automorphism of G that preserves colours is the identity. There are many classes of graphs for which the distinguishing chromatic number has been studied, including Cartesian products of complete graphs (Jerebic and Klavžar, 2010). In this paper we determine the distinguishing chromatic number of the complement of the Cartesian product of complete graphs, providing an interesting class of graphs, some of which have distinguishing chromatic number equal to the chromatic number, and others for which the difference between the distinguishing chromatic number and chromatic number can be arbitrarily large.

1. Introduction

A *proper colouring* of a graph, G, is a function that assigns labels (called colours) to the vertices of G so that adjacent vertices receive different colours. Since our interest is confined to proper colourings, we use the term *colouring* to mean proper colouring. The *chromatic number* of G, denoted $\chi(G)$, is the fewest number of colours required to colour G. Collins and Trenk [8] introduce the notion of distinguishing colouring and the distinguishing chromatic number. A *distinguishing colouring* of G is a colouring with the property that no automorphism of G, other than the identity, preserves the colours. The minimum number of colours required for a distinguishing colouring of G is called the *distinguishing chromatic number* of G, and is denoted by $\chi_D(G)$. Unless otherwise specified, we use the notation and terminology of [2].

Variants of graph colouring are ubiquitous in the literature, and distinguishing colouring is no exception; the appeal of graph colouring and symmetry-breaking has attracted much attention. The distinguishing chromatic number has been studied for various classes of graphs including paths, cycles, generalized Petersen graphs, hypercubes, interval graphs, Kneser graphs, line graphs, planar graphs, trees, and bipartite graphs (see [4,5,8,15–20]). Graphs with large distinguishing chromatic number are studied in [3], while a Nordhaus–Gaddum type theorem is given in [9]. The distinguishing chromatic number has also been investigated for graph operations such as Cartesian products, graph joins, wreath products, and other binary operations on graphs (see [6–8,10,14]). The related notion of distinguishing number is also well studied; results concerning Cartesian products can be found in [11,13].

In a 2010 paper, Choi, Hartke, and Kaul [6] investigate distinguishing chromatic numbers of Cartesian products of graphs, and determine many such classes of graphs for which the distinguishing chromatic number is either equal to the chromatic number, or at most one larger than the chromatic number. Jerebic and Klavžar [14] prove that if H is the Cartesian product of two complete graphs, then $\chi_D(H) = \chi(H)$, with only three exceptions, answering a question of Choi et al. [6] about graphs

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with nontrivial automorphisms for which the distinguishing chromatic number is equal to the chromatic number. These authors' interests are in finding graphs for which the chromatic number and the distinguishing chromatic number are close together.

The results in this paper are motivated, in part, by looking at the opposite end of the spectrum, i.e., graphs for which the chromatic number and the distinguishing chromatic number are far apart. As shown by Collins and Trenk [8], this difference can be made arbitrarily large since the distinguishing chromatic number of a complete multipartite graph is equal to the number of vertices in the graph. Cavers and Seyffarth [3] characterize all graphs for which the distinguishing chromatic number is one less or two less than the number of vertices. While studying distinguishing chromatic numbers of bipartite graphs, Laflamme and Seyffarth [17] show that the *complement* of the Cartesian product of K_2 with K_n, $n \geq 3$, has distinguishing chromatic number $\lceil 2\sqrt{n}\rceil$ (while the chromatic number is equal to two). The significance of this example is that the complement of the Cartesian product of K_2 with K_n is isomorphic to the complete bipartite graph $K_{n,n}$ minus a perfect matching, so is, in a sense "close" to being a complete multipartite graph (in this case bipartite). This led us to the problem of determining distinguishing chromatic numbers of complements of Cartesian products of complete graphs.

Recall that the *Cartesian product* of graphs G and H, denoted by $G\Box H$, has vertex set $V(G)\times V(H)$, with vertices (g,h) and (g',h') adjacent if and only if they are equal in one coordinate and adjacent in the other. In the current paper, we study complements of Cartesian products of complete graphs, and extend the result given in [17], that $\chi_D(\overline{K_2\Box K_n}) = \lceil 2\sqrt{n}\rceil$, by determining $\chi_D(\overline{\Box_{i=1}^n K_{a_i}})$ for $a_i \geq 2$ (see Theorem 21 and Corollary 22). The proof of our main result about the value of $\chi_D(\overline{\Box_{i=1}^n K_{a_i}})$ leads us to consider an optimization problem that possibly has other applications: given positive integers $p, q, b_1, \ldots, b_p$ such that $b_1 \times \cdots \times b_p \geq q$, find the minimum value of $b_1 + \cdots + b_p$.

2. Preliminaries

Let $n \geq 2$ be an integer, and let $a_1, \ldots, a_n$ be integers greater than one. Define $\mathcal{K} = \overline{\Box_{i=1}^n K_{a_i}}$, with

$$V(\mathcal{K}) = \{(u_1,\ldots,u_n) \mid 0 \leq u_i \leq a_i - 1 \text{ for } 1 \leq i \leq n\}.$$

Note that $\overline{\mathcal{K}}$ is called a Hamming graph [12]. Given a graph G and $X, Y \in V(G)$, we use the notation $X \sim Y$ if and only if $XY \in E(G)$. For $X, Y \in V(\mathcal{K})$, $X = (x_1,\ldots,x_n)$ and $Y = (y_1,\ldots,y_n)$, $X \sim Y$ in $\mathcal{K}$ if and only if the Hamming distance between strings $x_1\ldots x_n$ and $y_1\ldots y_n$ is at least two. Without loss of generality, assume $2 \leq a_1 \leq \cdots \leq a_n$, and let $a = \prod_{i=1}^n a_i = |V(\mathcal{K})|$. Let $[n] = \{1,2,\ldots,n\}$. For $U \in V(\mathcal{K})$ and $i \in [n]$, $U[i]$ denotes the ith coordinate of U.

Definition 1. Let $X = (x_1,\ldots,x_n) \in V(\mathcal{K})$, and let $i \in [n]$. An *i-row containing* X is a subset $R_{i,X} = \{(x_1,\ldots,x_{i-1},w,x_{i+1},\ldots,x_n) \mid 0 \leq w \leq a_i - 1\}$ of $V(\mathcal{K})$. Notice that if $Y \in R_{i,X}$, then $R_{i,Y} = R_{i,X}$. An *i-row* is an i-row containing X for some $X \in V(\mathcal{K})$.

The subgraph of $\overline{\mathcal{K}}$ induced by $R_{i,X}$ is called a *fiber* in [13]. Notice that for each $i \in [n]$ and each $X \in V(\mathcal{K})$, $R_{i,X}$ is an independent set of $\mathcal{K}$, and every independent set is a subset of an i-row. Thus $\alpha(\mathcal{K})$, the independence number of $\mathcal{K}$, is equal to a_n. It also follows that if a colouring of $\mathcal{K}$ assigns the same colour to vertices X and Y, then X and Y must belong to a common i-row for some $i \in [n]$.

Proposition 2. *A labelling f of $\mathcal{K}$ is a colouring of $\mathcal{K}$ if and only if for all $X, Y \in V(\mathcal{K})$, with $f(X) = f(Y)$, there exists an $i \in [n]$ such that $R_{i,X} = R_{i,Y}$.*

Proof. Let $f : V(\mathcal{K}) \to \{0,\ldots,k-1\}$ be a labelling of $\mathcal{K}$.

Suppose that f is a colouring of $\mathcal{K}$ and $X, Y \in V(\mathcal{K})$ are distinct vertices such that $f(X) = f(Y)$. Then $X \not\sim Y$; hence X and Y differ in exactly one coordinate, say $i \in [n]$. Thus $X[i] \neq Y[i]$ but $X[j] = Y[j]$ for all $j \in [n], j \neq i$. Therefore X and Y are in the same i-row, so $R_{i,X} = R_{i,Y}$.

For the converse, suppose that f is not a colouring of $\mathcal{K}$. Then there exist $X, Y \in V(\mathcal{K})$ such that $f(X) = f(Y)$ and $X \sim Y$. Since $X \sim Y$, X and Y differ in at least two coordinates. Thus for each $i \in [n]$, $Y \notin R_{i,X}$, and hence $R_{i,X} \neq R_{i,Y}$ for every $i \in [n]$. □

For a colouring f of $\mathcal{K}$, we denote by $\mathcal{C}_{f,j}$ the subset of $V(\mathcal{K})$ assigned colour j, and we say that $\mathcal{C}_{f,j}$ is a *colour class* of $\mathcal{K}$. When no ambiguity arises, we write $\mathcal{C}_j$ instead of $\mathcal{C}_{f,j}$.

Corollary 3. $\chi(\mathcal{K}) = a/a_n$.

Proof. Since $\alpha(\mathcal{K}) = a_n$, if follows that $\chi(\mathcal{K}) \geq a/a_n$. Since each n-row is an independent set and the n-rows partition $V(\mathcal{K})$, a colouring of $\mathcal{K}$ is $\phi : V(\mathcal{K}) \to \{0,\ldots,a/a_n - 1\}$ in which each n-row constitutes a distinct colour class. Thus, $\chi(\mathcal{K}) = a/a_n$. □

Definition 4. A *canonical colouring* of $\mathcal{K}$ is a colouring $\phi : V(\mathcal{K}) \to \{0,\ldots,a/a_n - 1\}$ in which each n-row constitutes a distinct colour class.

Fig. 1 depicts $\overline{\mathcal{K}} = K_2\Box K_3\Box K_4$. The 1-rows, 2-rows and 3-rows of $\mathcal{K} = \overline{K_2\Box K_3\Box K_4}$ correspond to, respectively, the K_2s with bold edges, K_3s with regular edges, and the K_4s with dashed edges. The labels zero through five on the 3-rows indicate the colour classes of a canonical colouring of $\mathcal{K}$.