Discrete Mathematics 341 (2018) 2442–2447

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# A reduction for block-transitive triple systems

Xiaoqin Zhan *, Suyun Ding

*School of Science, East China JiaoTong University, Nanchang, 330013, PR China*

ARTICLE INFO

*Article history:*
Received 28 October 2017
Received in revised form 12 April 2018
Accepted 22 May 2018

*Keywords:*
Block-transitive
Point-primitive
Automorphism group
Triple system

ABSTRACT

The paper reports on an investigation of structure of block-transitive automorphism groups of a triple system. We prove that if $G$ is a block-transitive automorphism group of a triple system $\mathcal{T}$, then $G$ cannot be of simple diagonal or twisted wreath product type. Furthermore, if $G$ is of product type, then $\mathrm{Soc}(G) = A_5 \times A_5$ and $\mathcal{T}$ is the unique $TS(25, 12)$.



## 1. Introduction

A 2-$(v, k, \lambda)$ *design* $\mathcal{D}$ is an incidence structure $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of $v$ points and $\mathcal{B}$ is a set of $b$ blocks with incidence relation such that every block is incident with exactly $k$ points and every 2-element subset of $\mathcal{P}$ is incident with exactly $\lambda$ blocks. Let $r$ be the number of blocks incident with a given point. A *linear space* $\mathcal{S}$ is a 2-$(v, k, 1)$ design. In $\mathcal{S}$, a block is usually called a line. A *triple system* $TS(v, \lambda)$ is a 2-$(v, 3, \lambda)$ design, denoted by $\mathcal{T}$. It is well known that a necessary and sufficient condition for the existence of a $TS(v, \lambda)$ is $v \neq 2$ and $\lambda \equiv 0 \,(\mathrm{mod}\,(v-2, 6))$.

An *automorphism* of a $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The *full automorphism group* of $\mathcal{D}$ consists of all automorphisms of $\mathcal{D}$ and is denoted by $\mathrm{Aut}(\mathcal{D})$. A subgroup $G$ of the automorphism group of $\mathcal{D}$ is *block-transitive* if it acts transitively on $\mathcal{B}$; $\mathcal{D}$ is block-transitive if $\mathrm{Aut}(\mathcal{D})$ is. Point- and flag-transitivity are defined similarly, where a *flag* of $\mathcal{D}$ is an incident point-block pair $(\alpha, B)$. A set of blocks of incidence structure $\mathcal{D}$ is called a set of *base blocks* with respect to an automorphism group $G$ of $\mathcal{D}$ if it contains exactly one block from each $G$-orbit on the block set. In particular, if $G$ is a block-transitive automorphism group of $\mathcal{D}$, then any block $B$ is a base block of $\mathcal{D}$.

Most research on block-transitive 2-$(v, k, \lambda)$ designs hitherto has concerned the case $\lambda = 1$. In 1984 Camina and Gagen showed in [4] that if $G$ acts transitively on the blocks of a 2-$(v, k, 1)$ design and if $k$ divides $v$, then $G$ is flag-transitive, hence point-primitive by a result of Higman and McLaughlin [12]. Inspired by the proof, several others [2,7,17] generalized the result in [4] to prove that groups acting flag-transitively on linear spaces are affine or almost simple. It is worth nothing that both [7] and [17] generalized the result to the situation of 2-designs with $(r, \lambda) = 1$ (the greatest common divisor of $r$ and $\lambda$ is 1).

Several years after these results Camina and Praeger showed in [5] that if $G$ acts block-transitively on a linear space $\mathcal{S}$, then $G$ has at most one non-abelian minimal normal subgroup. Then in [3] Camina presented that if $G$ is block-transitive and point-primitive on $\mathcal{S}$, then $G$ is affine or almost simple.

Let $\mathcal{T}$ be a triple system admitting a block-transitive automorphism group $G$. According to a result of Block [1], $G$ is also point-transitive. By Lemma 2.2, $G$ acts primitively on the points of $\mathcal{T}$. particularly, if $G$ acts flag-transitively on $\mathcal{T}$, then $G$ is an affine or almost simple group (see Lemma 2.3).

* Corresponding author.
*E-mail address:* zhanxiaoqinshuai@126.com (X. Zhan).

https://doi.org/10.1016/j.disc.2018.05.021

The aim of this paper is to show that an automorphism group $G$ which acts transitively on the blocks of $\mathcal{T}$ cannot be of simple diagonal or twisted wreath product type.

We now state the theorem we intend to prove.

**Theorem 1.** *Let $G$ acts as a block-transitive automorphism group of a triple system $\mathcal{T}$. Then one of the following holds:*

(1) *$G$ is of affine type.*
(2) *$G$ is of almost simple type.*
(3) *$G$ is of product type with $\mathrm{Soc}(G) = A_5 \times A_5$, and $\mathcal{T}$ is the unique $TS(25, 12)$.*

The paper is organized as follows. In Section 2, we introduce some preliminary results that are important for the remainder of the paper. In Section 3, we complete the proof of Theorem 1 in three parts.

## 2. Preliminaries

The notation and terminology used is standard and can be found in [6,10] for design theory and in [11,14,15] for group theory. In particular, if $G$ is a permutation group on a set $\Omega$, and $\alpha \in B \subseteq \Omega$, then $G_\alpha$ denotes the stabilizer in $G$ of a point $\alpha$, and $G_B$ denotes the setwise stabilizer in $G$ of a set $B$, and $\mathrm{Soc}(G)$ denotes the socle of $G$.

**Lemma 2.1** ([6, II 1.2, 1.9]). *The parameters $v, b, r, \lambda$ of a triple system satisfy the following conditions:*

(i) $vr = 3b$.
(ii) $\lambda(v-1) = 2r$.
(iii) $b = \frac{\lambda v(v-1)}{6} \geq v$.

**Lemma 2.2.** *If $\mathcal{T}$ is a triple system admitting a block-transitive automorphism group $G$, then $G$ is point-primitive.*

**Proof.** By the result of Delandtsheer and Doyen [9, Theorem], we have that a block-transitive, point-imprimitive 2-$(v, k, \lambda)$ design satisfies

$$v \leq (\binom{k}{2} - 1)^2.$$

Since the number of points of $\mathcal{T}$ is at least 5, the primitivity of $G$ is obvious. □

**Lemma 2.3.** *If $G$ is a flag-transitive automorphism group of a triple system $\mathcal{T}$, then $G$ acts 2-homogeneously on the point set $\mathcal{P}$. Moreover, $G$ is an affine or almost simple group.*

**Proof.** Let $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ be two arbitrary unordered pairs of $\mathcal{P}$. By the definition of a triple system, there are two points $\varepsilon$ and $\theta$ such that $B_1 = \{\alpha, \beta, \varepsilon\}$ and $B_2 = \{\gamma, \delta, \theta\}$ are two blocks of $\mathcal{T}$. The flag-transitivity of $G$ implies that there is a $g \in G$ such that

$$(\varepsilon, B_1)^g = (\varepsilon^g, B_1^g) = (\theta, B_2),$$

and so $\{\alpha, \beta\}^g = \{\gamma, \delta\}$. Thus $G$ is 2-homogeneous and the result follows from [11, Theorem 4.1B, Theorem 9.4B(i)]. □

The following lemma is well-known.

**Lemma 2.4.** *Let $\mathcal{D}$ be a 2-$(v, k, \lambda)$ design and $G$ be an automorphism group of $\mathcal{D}$. For any point $\alpha \in \mathcal{P}$ and block $B \in \mathcal{B}$, then $G$ acts flag-transitively on $\mathcal{D}$ if and only if*

(i) *$G$ acts transitively on $\mathcal{P}$ and $G_\alpha$ acts transitively on the pencil $P(\alpha)$ (the set of blocks through $\alpha$), or*
(ii) *$G$ acts transitively on $\mathcal{B}$ and $G_B$ acts transitively on the points of $B$.*

**Lemma 2.5.** *Let $\mathcal{T}$ be a triple system with a block-transitive automorphism group $G$. Then $r$ divides $3|G_\alpha|$. Furthermore, $r$ divides $3\lambda d$, for all non-trivial subdegrees $d$ of $G$.*

**Proof.** Let $B$ be a block of $\mathcal{T}$ containing the point $\alpha$. Then

$$[G : G_{\alpha B}] = [G : G_\alpha][G_\alpha : G_{\alpha B}] = v[G_\alpha : G_{\alpha B}].$$

But

$$[G : G_{\alpha B}] = [G : G_B][G_B : G_{\alpha B}] = \frac{vr}{3}[G_B : G_{\alpha B}].$$

So $[G_\alpha : G_{\alpha B}] = \frac{r[G_B : G_{\alpha B}]}{3}$ and $r \mid 3|G_\alpha|$. Clearly, $[G_B : G_{\alpha B}] = 1, 2$ or $3$ as $|B| = 3$.