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Note

The maximum girth and minimum circumference of graphs with prescribed radius and diameter

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ABSTRACT

Ostrand posed the following two questions in 1973. (1) What is the maximum girth of a graph with radius r and diameter d? (2) What is the minimum circumference of a graph with radius r and diameter d? Question 2 has been answered by Hrnčiar who proves that if $d \leq 2r-2$ the minimum circumference is $4r-2d$. In this note we first answer Question 1 by proving that the maximum girth is $2r+1$. This improves on the obvious upper bound $2d+1$ and implies that every Moore graph is self-centered. We then prove a property of the blocks of a graph which implies Hrnčiar's result.

1. Introduction

We consider finite simple graphs. Ostrand [6, p.75] posed the following two questions in 1973.

Question 1. What is the maximum girth of a graph with radius r and diameter d?

Question 2. What is the minimum circumference of a graph with radius r and diameter d?

Question 2 has been answered by Hrnčiar [5] who proves that if $d \leq 2r-2$ the minimum circumference is $4r-2d$. In this note we first answer Question 1 by proving that the maximum girth is $2r+1$. This improves on the obvious upper bound $2d+1$ and implies that every Moore graph is self-centered. We then prove a property of the blocks of a graph which implies Hrnčiar's result.

Google shows 63 citations of Ostrand's paper [6] and MathSciNet shows 7 citations. It seems that Question 1 has not been treated.

For terminology and notations we follow the books [1,3,8]. We denote by $V(G)$ the vertex set of a graph G and by $d(u, v)$ the distance between two vertices u and v. The *eccentricity*, denoted by $e(v)$, of a vertex v in a graph G is the distance to a vertex farthest from v. Thus $e(v) = \max\{d(v, u)|u \in V(G)\}$. If $e(v) = d(v, x)$, then the vertex x is called an *eccentric vertex* of v. The *radius* of a graph G, denoted $\mathrm{rad}(G)$, is the minimum eccentricity of all the vertices in $V(G)$, whereas the diameter of G, denoted $\mathrm{diam}(G)$, is the maximum eccentricity. A vertex v is a *central vertex* of G if $e(v) = \mathrm{rad}(G)$. When H is a subgraph of a graph G and $u, v \in V(H)$, $d_H(u, v)$ and $e_H(v)$ will mean the distance and eccentricity in H respectively.

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2. Main results

Recall that a *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. Thus in a connected graph, a block is either a cut-edge or a maximal 2-connected subgraph of order at least 3. If H is a subgraph of a graph G and v is a vertex of G, then the distance between v and H, denoted $d(v, H)$, is defined as $d(v, H) = \min\{d(v, x)|x \in V(H)\}$. Throughout the symbol C_k denotes a cycle of length k, and P_t denotes a path of order t (and hence of length $t - 1$).

Lemma 1. *If B is a block of a graph G, then* $\mathrm{rad}(B) \le \mathrm{rad}(G)$.

Proof. Let c be a central vertex of G. If $c \in V(B)$, then $\mathrm{rad}(B) \le e_B(c) \le e_G(c) = \mathrm{rad}(G)$. Thus $\mathrm{rad}(B) \le \mathrm{rad}(G)$. Now suppose $c \notin V(B)$. In the block-cutvertex tree of G ([1, p.121] or [8, p.156]), from any block containing c there is a unique path to B. Hence there is a unique vertex v of B such that $d(c, B) = d(c, v)$. Note that v is a cut-vertex of G. Let u be an eccentric vertex of v in B; i.e., $e_B(v) = d(v, u)$. Then we have

$$\begin{aligned}\mathrm{rad}(G) = e_G(c) \ge d(c, u) &= d(c, v) + d(v, u)\\ &\ge 1 + e_B(v) \ge 1 + \mathrm{rad}(B),\end{aligned}$$

showing that $\mathrm{rad}(G) > \mathrm{rad}(B)$ in this case. □

We remark that in Lemma 1, if we replace the block B by a generic subgraph, the conclusion may not be true. To see this, consider the wheel graph W_n of order $n \ge 5$ with central vertex c. We have $\mathrm{rad}(W_n - c) = \lfloor (n-1)/2 \rfloor > 1 = \mathrm{rad}(W_n)$.

We make the convention that the girth of an acyclic graph is undefined. Thus whenever we speak of the girth of a graph, we have already implicitly assumed that the graph contains at least one cycle. We will need the well-known fact (e.g. [6]) that if r and d are the radius and diameter of a graph respectively, then $r \le d \le 2r$. The following result answers Question 1.

Theorem 2. *The maximum girth of a graph with radius r and diameter d is* $2r + 1$.

Proof. Let G be a graph with radius r and diameter d containing at least one cycle. Every cycle lies within one block. Let B be a block of G containing a cycle. Then B is 2-connected. Let x be a central vertex of the subgraph B. There is at least one cycle containing x, since B is 2-connected [8, p.162]. Among all the cycles containing x, we choose one of the shortest length and denote it by C. Let $s = \mathrm{rad}(B)$ and denote the length of C by q. We assert that $q \le 2s+1$. To the contrary suppose $q \ge 2s+2$. Since C is a shortest cycle containing x, C contains a vertex y such that

$$d_B(x, y) = d_C(x, y) = \lfloor q/2 \rfloor \ge s + 1,$$

implying that $s = e_B(x) \ge d_B(x, y) \ge s + 1$, a contradiction. By Lemma 1, $s \le r$. We deduce that G has girth at most q and $q \le 2s + 1 \le 2r + 1$.

Conversely, given any positive integers r and d with $r \le d \le 2r$, let $M(r, d)$ be the monocle graph which is the union of the cycle C_{2r+1} and the path P_{d-r+1}, the cycle and the path having only one common vertex which is an end vertex of the path. Then $M(r, d)$ has radius r, diameter d and girth $2r + 1$. This completes the proof. □

Theorem 2 improves on the obvious upper bound $2d+1$. A *Moore graph* was originally defined in [4] as a graph of diameter d, maximum degree Δ and the largest possible order $1 + \Delta\sum_{i=1}^{d}(\Delta - 1)^{i-1}$. An equivalent definition [2,7] of a Moore graph is a graph of diameter d and girth $2d + 1$. A graph G is said to be *self-centered* if $\mathrm{rad}(G) = \mathrm{diam}(G)$. Thus self-centered graphs are those graphs in which every vertex is a central vertex. The following result is deduced in [3, pp.100–101] by using the concept of the distance degree sequence of a vertex. Now it becomes obvious.

Corollary 3. *Every Moore graph is self-centered.*

Proof. Let G be a Moore graph of radius r, diameter d and girth $2d + 1$. By Theorem 2 and the fact $r \le d$ we have $2d + 1 \le 2r + 1 \le 2d + 1$. Hence $r = d$. □

Now we prove a property of the blocks of a graph. In the proof we will use an idea in [5].

Theorem 4. *Every graph of radius r and diameter d has a block whose diameter is at least* $2r - d$.

Proof. We first prove the following

Claim. *Let H be a graph of radius r and diameter d. If B is a block of H with* $\mathrm{diam}(B) < 2r - d$ *and u is a vertex of H such that* $d(u, B) = \max\{d(x, B)|x \in V(H)\}$, *then* $d(u, B) \ge r$.

In the block-cutvertex tree of H ([1, p.121] or [8, p.156]), from the block (necessarily an end block) containing u to B there is a unique path. Thus there is a unique vertex $v \in V(B)$ such that $d(u, B) = d(u, v)$ and v is a cut-vertex of H. Denote $d(u, v) = a$. We need to prove $a \ge r$. To the contrary, suppose $a < r$. We will show that $e(v) < r$, which is a contradiction