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Bijections between generalized Catalan families of types A and C

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ABSTRACT

Motivated by the relation $\mathrm{N}^m(C_n) = (mn+1)\mathrm{N}^m(A_{n-1})$, holding for the m-generalized Catalan numbers of type A and C, the connection between dominant regions of the m-Shi arrangement of type A_{n-1} and C_n is investigated. More precisely, it is explicitly shown how $mn+1$ copies of the set of dominant regions of the m-Shi arrangement of type A_{n-1}, biject onto the set of type C_n such regions. This is achieved by exploiting two different viewpoints of the representative alcove of each region: the Shi tableau and the abacus diagram. In the same line of thought, a bijection between $mn+1$ copies of the set of m-Dyck paths of height n and the set of $N-E$ lattice paths inside an $n \times mn$ rectangle is provided.

1. Introduction

The classical Catalan numbers, $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$, constitute one of the most ubiquitous number sequences in enumerative combinatorics, also appearing in several other contexts varying from algebra and representation theory [13,14,21,22] to discrete geometry [3,8,20]. We refer to [24] for a list of 214 families of objects enumerated by $\mathrm{Cat}(n)$. Such objects (also called *Catalan objects*) relevant to the present paper are the set of dominant regions of the Shi arrangement $\mathrm{Shi}(A_{n-1})$, the triangulations of a convex $(n+3)$-gon, and the Dyck paths of length n. In [3], Athanasiadis generalized Catalan numbers for every crystallographic root system Φ and positive integer m. More precisely, he defined the *m-generalized Catalan* number of type Φ as

$$\mathrm{N}^m(\Phi) = \prod_{i=1}^{n} \frac{e_i + mh + 1}{e_i + 1}, \tag{1.1}$$

where n is the rank, h is the Coxeter number and e_i are the exponents of Φ. In particular, he showed that $\mathrm{N}^m(\Phi)$ counts the number of *dominant regions* of the *m-Shi arrangement* associated to Φ (see Section 1.1 for the undefined notions). We note that the classical Catalan numbers $\mathrm{Cat}(n)$ are indeed a special case of (1.1), since they occur when $\Phi = A_{n-1}$ and $m = 1$. More generally, if $\Phi = A_{n-1}$ or C_n and m is an arbitrary positive integer, the expression in (1.1) reduces respectively to

$$\mathrm{N}^m(A_{n-1}) = \frac{1}{mn+1}\binom{(m+1)n}{n} \qquad \text{and} \qquad \mathrm{N}^m(C_n) = \binom{(m+1)n}{n}.$$

The numbers $\mathrm{N}^m(A_{n-1})$, also known as *Fuss–Catalan numbers*, count a wealth of combinatorial objects, most of which can be seen as m-generalizations of *type A* Catalan objects. For the families discussed above i.e., dominant regions of Shi

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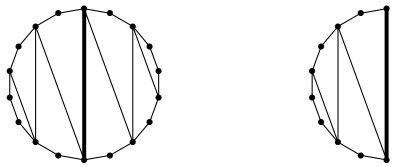
arrangements, polygon triangulations and Dyck paths, such m-generalizations have been an object of research for more than a decade. In the same spirit generalized (m, Φ)-Catalan objects have been discovered for every finite root system Φ and integer m (see for instance [1,3,13]). In most cases each root system Φ was studied separately before a unified structure was discovered. Almost always the starting point was the relation

$$\mathsf{N}^m(A_{n-1})(mn+1) = \mathsf{N}^m(C_n), \tag{1.2}$$

holding between m-Catalan numbers of type A and C. Occasionally, an (m, C_n)-Catalan object is a type A one, of certain size and symmetry, where $mn+1$ copies of the (m, A_{n-1})-Catalan object reside. Although most of the times the symmetry is rather natural to guess or understand, locating the $mn+1$ copies of the (m, A_{n-1})-Catalan object in the corresponding (m, C_n)-type can vary from easy to very complicated.

To give a motivating example we describe a class of (m, Φ)-Catalan objects where Relation (1.2) arises trivially. Consider the set $\mathscr{D}_n^m$ of $(m+2)$-angulations of a convex $(mn+2)$-gon P i.e., dissections of P by noncrossing diagonals into polygons each having $m+2$ vertices. Let also $\mathscr{C}_{2n}^m$ be the subset of $\mathscr{D}_{2n}^m$ consisting of centrally symmetric $(m+2)$-angulations of a $(2mn+2)$-gon. The sets $\mathscr{D}_n^m$ and $\mathscr{C}_{2n}^m$ are combinatorial realizations of the facets of the generalized cluster complex $\Delta^m(\Phi)$ for $\Phi = A_{n-1}$ and C_n respectively [13]. From their description Relation (1.2) is evident: we identify each centrally symmetric



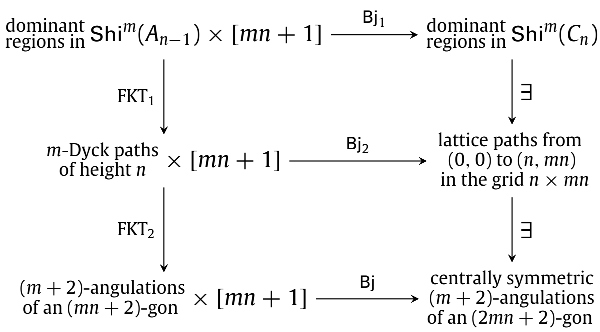
dissection D in $\mathscr{C}_{2n}^m$ with the pair consisting of its diameter and one copy of the two $(m+2)$-angulations of the $(mn+2)$-gon into which the diameter divides the initial $(2mn+2)$-gon. Since the diameters are $mn+1$, the cardinality of $\mathscr{C}_{2n}^m$ is indeed $(mn+1)\mathsf{N}^m(A_{n-1}) = \mathsf{N}^m(C_n)$.

The aim of this work is to reveal two new instances of the identity $\mathsf{N}^m(A_{n-1})(mn+1) = \mathsf{N}^m(C_n)$, which might lead to a better geometric understanding of the type A and C Shi arrangements. More precisely, we provide an *explicit* bijection between each of the following pair of sets:

(Bj_1) the set containing $mn+1$ copies of each dominant region of the m-Shi arrangement $\mathsf{Shi}^m(A_{n-1})$ and that of dominant regions in $\mathsf{Shi}^m(C_n)$

(Bj_2) the set containing $mn+1$ copies of each m-Dyck path of height n and that of lattice paths from $(0, 0)$ to (n, mn) in the grid $n \times mn$.

We note that the second bijection is based on an idea in [19] while the first, which is the main contribution of this work, relies on the two different ways to view the representative alcove of a region in $\mathsf{Shi}^m(\Phi)$: its *Shi tableau* [10] and its *abacus diagram* [2,12]. Each of the bijections Bj_1,Bj_2 can stand on its own and the sections presenting them (Sections 2 and 3 respectively) can be read independently. However, in the setting of dominant regions in $\mathsf{Shi}^m(A_n)$, there exists previous work [10] which reveals a connection between the two bijections. Unifying previous and current results we have the following commutative diagram:



where FKT_1 and FKT_2 are bijections given in [10].

This paper is structured as follows. We end up this section by recalling basic facts on root systems and presenting the two families of Catalan objects we are dealing with. In Section 2 we introduce all necessary material and build our first bijection Bj_1. More precisely, in Section 2.1 we include background on Shi arrangements, discuss the notion of *m-minimal alcoves* and explain their connection to dominant regions in $\mathsf{Shi}^m(\Phi)$. Subsequently, in Sections 2.2 and 2.3, we describe the two different viewpoints in which we can encode dominant alcoves: Shi tableaux and abacus diagrams. Section 2.4 serves to clarify the relation between them in the type A case and to show how this adjusts to the type C case. Section 2.5 provides criteria for m-minimality in terms of the abacus representation. Finally, in Section 2.6, we construct the bijection Bj_1 while