Discrete Mathematics 341 (2018) 2841–2845

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A combinatorial characterisation of embedded polar spaces

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ARTICLE INFO

Article history:
Received 21 August 2017
Received in revised form 20 April 2018
Accepted 27 June 2018

Keywords:
Finite classical polar space

ABSTRACT

Some classical polar spaces admit polar spaces of the same rank as embedded polar spaces (often arisen as the intersection of the polar space with a non-tangent hyperplane). In this article we look at sets of generators that behave combinatorially as the set of generators of such an embedded polar space, and we prove that they are the set of generators of an embedded polar space.

1. Introduction

A fundamental question in finite geometry is to recognise geometric substructures from combinatorial properties. One of the first questions of this kind was posed (and solved) by Beniamino Segre, who showed that an *oval* (which is defined in a combinatorial way) of a finite Desarguesian projective plane of odd order, is necessarily a *conic* (which is a non-singular curve of degree two). Comparable questions have been studied in higher dimensional projective spaces and also in finite classical polar spaces, of which the following theorem is an example.

Theorem 1.1 (*De Winter and Schillewaert, [3]*). *If a point set K in $\mathrm{PG}(n,q)$, $n \geq 4$, $q > 2$, has the same intersection numbers with respect to hyperplanes and subspaces of codimension 2 as a polar space $\mathcal{P} \in \{\mathcal{H}(n,q^2), \mathcal{Q}^+(2n+1,q), \mathcal{Q}^-(2n+1,q), \mathcal{Q}(2n,q)\}$, then K is the point set of a non-singular polar space $\mathcal{P}$.*

In this paper, we deal with a rather particular situation in finite classical polar spaces. Motivated by research in [2], the aim is to recognise embedded finite classical polar spaces as sets of generators of a larger polar space satisfying some combinatorial properties. As such, we want to provide a proof of Theorem 6.6 in [2], and this is the main aim of this paper.

Polar spaces were introduced in an axiomatic way by Veldkamp [7,8] and Tits [6]. A *polar space* is a point-line geometry satisfying the one-or-all axiom, i.e. for a given point P and a given line l not through P, the point P is collinear with one point of l or with all points of l. This characterisation is due to Buekenhout and Shult. Although this remarkable characterisation is often very useful in geometrical and combinatorial proofs of theorems on polar spaces, we will prefer the use of the original definition of polar spaces of Tits, which turns out to make our proofs shorter.

Definition 1.2. A *polar space of rank d*, $d \geq 3$, is an incidence geometry (Π, Ω) with Π a set whose elements are called points and Ω a set of subsets of Π satisfying the following axioms.

(1) Any element $\omega \in \Omega$ together with the elements of Ω that are contained in ω, is a projective geometry of dimension at most $d-1$.

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https://doi.org/10.1016/j.disc.2018.06.040

Table 1
Different flavours of finite classical polar spaces, parameter and rank.

Family	Subfamily	Notation	Ambient space	Rank	Parameter
Orthogonal	Elliptic	$\mathcal{Q}^-(2n+1,q)$	$\mathrm{PG}(2n+1,q), n \geq 1$	n	2
	Parabolic	$\mathcal{Q}(2n,q)$	$\mathrm{PG}(2n,q), n \geq 1$	n	1
	Hyperbolic	$\mathcal{Q}^+(2n+1,q)$	$\mathrm{PG}(2n+1,q), n \geq 0$	$n+1$	0
Hermitian	Odd dimension	$\mathcal{H}(2n+1,q^2)$	$\mathrm{PG}(2n+1,q^2), n \geq 0$	$n+1$	1/2
	Even dimension	$\mathcal{H}(2n,q^2)$	$\mathrm{PG}(2n,q^2), n \geq 1$	n	3/2
Symplectic		$\mathcal{W}(2n+1,q)$	$\mathrm{PG}(2n+1,q), n \geq 0$	$n+1$	1

(2) The intersection of two elements of Ω is an element of Ω (the set Ω is closed under intersections).
(3) For a point $P \in \Pi$ and an element $\omega \in \Omega$ of dimension $d-1$ such that P is not contained in ω there is a unique element $\omega' \in \Omega$ of dimension $d-1$ containing P such that $\omega \cap \omega'$ is a hyperplane of ω. The element ω is the union of all 1-dimensional elements of Ω that contain P and are contained in ω.
(4) There exist two elements Ω both of dimension $d-1$ whose intersection is empty.

One of the consequences of the theory developed in [6] is that all polar spaces of rank at least 3 arise from a sesquilinear or quadratic form acting on a vector space over a (skew) field. In the finite case, this means that finite polar spaces of rank at least 3 are known and classified. We assume that the reader is familiar with finite classical polar spaces. To fix the notation, we refer to Table 1, listing the six different families of finite classical polar spaces including rank and parameter.

The finite field of order q, $q = p^h$, p prime and $h \geq 1$, will be denoted as $\mathbb{F}_q$, and the n-dimensional projective space over $\mathbb{F}_q$ as $\mathrm{PG}(n,q)$.

A finite classical polar space of rank d over $\mathbb{F}_q$ has parameter $e = \log_q(x-1)$ with x the number of generators through a fixed $(d-2)$-space. The following lemma summarises the number of points and generators of a finite classical polar space using rank, parameter and order of the field. Recall that $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$ denotes the Gaussian coefficient representing the number of k-dimensional spaces in $\mathrm{PG}(n,q)$. Note that for $0 \leq m < r$ one defines $\begin{bmatrix} m \\ r \end{bmatrix}_q = 0$.

Lemma 1.3. *The number of generators of a finite classical polar space of rank d with parameter e, embedded in a projective space over $\mathbb{F}_q$, is given by $\prod_{i=0}^{d-1}(q^{e+i}+1)$. Its number of points equals $\begin{bmatrix} d \\ 1 \end{bmatrix}_q (q^{d+e-1}+1)$.*

The number of generators on a classical finite polar space of rank d with parameter e, embedded in a projective space over $\mathbb{F}_q$, through a fixed point is $\prod_{i=0}^{d-2}(q^{e+i}+1)$.

Consider a polar space $\mathcal{P}$ of rank d, defined over $\mathbb{F}_q$. Any hyperplane π of the ambient projective space which is not a tangent hyperplane to $\mathcal{P}$, contains or intersects the elements of $\mathcal{P}$. The elements completely contained in the hyperplane constitute a finite classical polar space $\mathcal{P}'$ in π. The polar space $\mathcal{P}'$ may be of the same rank as $\mathcal{P}$, but will have a different parameter. In this paper we are interested in the cases where the rank of $\mathcal{P}'$ equals the rank of $\mathcal{P}$. This restricts us to the following cases: $\mathcal{Q}^+(2n-1,q) \subset \mathcal{Q}(2n,q)$, $\mathcal{Q}(2n,q) \subset \mathcal{Q}^-(2n+1,q)$ and $\mathcal{H}(2n-1,q^2) \subset \mathcal{H}(2n,q^2)$.

Using the particular isomorphism between $\mathcal{Q}(2n,q)$ and $\mathcal{W}(2n-1,q)$, q even, also the embedding $\mathcal{Q}^+(2n+1,q) \subset \mathcal{W}(2n+1,q)$, q even, is known. Our result will also include this case.

Definition 1.4. Let $\mathcal{P}$ be a finite classical polar space of rank $d \geq 3$ and with parameter $e \geq 1$, embedded in a projective space over $\mathbb{F}_q$. A set $\mathcal{S}$ of generators of $\mathcal{P}$ is called *strong pseudopolar* if

(i) for every $i = 0, \ldots, d$ the number of elements of $\mathcal{S}$ meeting a generator π in a $(d-i-1)$-space equals

$$\begin{cases} \left(\begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q + q^i \begin{bmatrix} d-1 \\ i \end{bmatrix}_q\right) q^{\binom{i-1}{2}+ie-1} & \text{if } \pi \in \mathcal{S} \\ (q^{e-1}+1)\begin{bmatrix} d-1 \\ i-1 \end{bmatrix}_q q^{\binom{i-1}{2}+(i-1)e} & \text{if } \pi \notin \mathcal{S} \end{cases};$$

(ii) for every point P of $\mathcal{P}$ there is a generator $\pi \notin \mathcal{S}$ through P;
(iii) for every point P of $\mathcal{P}$ and every generator $\pi \notin \mathcal{S}$ through P, there are either $(q^{e-1}+1)\begin{bmatrix} d-2 \\ j \end{bmatrix}_q q^{\binom{j}{2}+je}$ generators of $\mathcal{L}$ through P meeting τ in a $(d-j-2)$-space, for all $j = 0, \ldots, d-2$, or there are no generators of $\mathcal{L}$ through P meeting τ in a $(d-j-2)$-space, for all $j = 0, \ldots, d-2$.

The aim of this paper is precisely to show a characterisation of polar spaces $\mathcal{P}'$ embedded in a polar space $\mathcal{P}$ of the same rank through the combinatorial and geometrical behaviour of the set of generators of $\mathcal{P}'$ as subset of the generators of $\mathcal{P}$. In other words, if a set of generators of a finite classical polar space behaves combinatorially as the set of generators of an embedded polar space of the same rank, is it really the set of generators of an embedded polar space? The main theorem of this paper is the following.