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Rainbow number of matchings in planar graphs

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ABSTRACT

The *rainbow number* $rb(G, H)$ for the graph H in G is defined to be the minimum integer c such that any c-edge-coloring of G contains a rainbow H. As one of the most important structures in graphs, the rainbow number of matchings has drawn much attention and has been extensively studied. Jendrol et al. initiated the rainbow number of matchings in planar graphs and they obtained bounds for the rainbow number of the matching kK_2 in the plane triangulations, where the gap between the lower and upper bounds is $O(k^3)$. In this paper, we show that the rainbow number of the matching kK_2 in maximal outerplanar graphs of order n is $n+O(k)$. Using this technique, we show that the rainbow number of the matching kK_2 in some subfamilies of plane triangulations of order n is $2n + O(k)$. The gaps between our lower and upper bounds are only $O(k)$.

1. Introduction

An edge-colored graph is called *rainbow* if the colors on its edges are distinct. The *rainbow number* $rb(K_n, H)$ for the graph H in K_n is defined to be the minimum integer k such that any k-edge-coloring of K_n contains a rainbow H. This parameter is closely related to the *anti-Ramsey number* for a graph H in K_n, introduced by Erdős et al. [4] in 1973, which is defined to be the maximum number of colors in an edge-coloring of K_n which does not contain any rainbow H. Clearly, the rainbow number equals the anti-Ramsey number plus one. Interestingly, during the last decades the researchers replaced the host graph K_n by a graph G and the corresponding rainbow number $rb(G, H)$ for a graph H in G, if $H \subseteq G$, can be defined in the similar way.

In general, the rainbow number is closely related to the Turán result. So far, the rainbow numbers for some special graph classes have been determined, see [5]. Among them, the most intriguing classes include cycles, paths, cliques and matchings. The rainbow numbers of paths and cliques in K_n have been well studied and determined in [4,24] and [4,20,23], respectively. Erdős et al. [4] posed a conjecture on the rainbow number of cycles in K_n, which was proved step by step in [1,9,21]. Axenovich et al. [2] determined the rainbow number of cycles in complete bipartite graphs. When the host graph is not K_n or $K_{m;n}$, the problem seems to be very complicated and hard to determine. Encouragingly, Hornak et al. [7] gave bounds for the rainbow number of cycles in plane triangulations, which was improved recently by Lan et al. [16]. For more results about the rainbow numbers, we refer to the surveys [5,15] and [6] which is a dynamically updated version of [5].

As one of the most important structures in graphs, the rainbow number of matchings has drawn much attention and has been extensively studied. It is not surprising that the rainbow numbers of matchings, including its extremal colorings, in K_n and $K_{m;n}$ have been determined completely, see [3,12,14,17,23]. Later, the authors [10,11,18] further considered the rainbow number of matchings in regular bipartite graphs. Interestingly, the authors [22] considered the rainbow number of matchings in hypergraphs. Recently, Jin et al. [13] determined the problem in complete split graphs which covers the results

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in complete graphs. Jendrol et al. [8] obtained bounds for the rainbow number of matchings in the plane triangulations. Unfortunately, the gap between the lower and upper bounds in [8] is $O(k^3)$, where k denotes the size of the matching.

In this paper, we consider the rainbow number of the matching kK_2 in planar graphs. At first in Section 2, using the analogous technique from [8], we get the lower and upper bounds for the rainbow number of matching kK_2 in maximal outerplanar graphs. Still the gap between them is somewhat large. In Section 3, we find a new method to improve the bounds in Section 2 and we show that the rainbow number of the matching kK_2 in maximal outerplanar graphs of order n is $n + O(k)$. Using this technique, we show that the rainbow number of the matching kK_2 in the Hamiltonian subfamily of plane triangulations of order n is $2n+O(k)$, which improves the bounds in [8]. Finally in Section 4, we determine the rainbow number of matchings of small size in the outerplanar graphs.

2. Bounds in maximal outerplanar graphs

A maximal outerplanar graph is a planar graph that is not a spanning subgraph of another outerplanar graph. Clearly, if M_n is a maximal outerplanar graph of order $n \geq 4$, then $|E(M_n)| = 2n - 3$ and $\delta(M_n) \geq 2$. Denote by $\mathcal{M}_n$ the family of maximal outerplanar graphs of order n. We also use the notion $rb(\mathcal{M}_n, kK_2)$ to denote the minimum integer k such that any k-edge-coloring of any member of $\mathcal{M}_n$ contains a rainbow kK_2. In this section, we give lower and upper bounds on $rb(\mathcal{M}_n, kK_2)$ for all $k \geq 3$ and $n \geq 2k$.

Lemma 2.1. *For all $k \geq 3$ and $n \geq 2k$, $rb(\mathcal{M}_n, kK_2) \geq n + 2k - 5$.*

Proof. Let M_n be a maximal outerplanar graph of order n with $V(M_n) = \{v_1, v_2, \dots, v_n\}$ and $E(M_n) = \{v_iv_{i+1} | 1 \leq i \leq n-1\} \bigcup \{v_1v_i | 3 \leq i \leq n\}$. It is clear that $rb(\mathcal{M}_n, kK_2) \geq rb(M_n, kK_2)$. Now color all edges v_iv_{i+1} for $2k-4 \leq i \leq n-1$ with a same color and color all the remaining edges with new distinct colors. Then we get a $(n + 2k - 6)$-edge-coloring of M_n, which does not contain any rainbow kK_2. So the lower bound holds obviously. ■

Now we consider the upper bound of rainbow matchings in the maximal outerplanar graph. First, we present an upper bound by using the same proof idea in [8].

Lemma 2.2. *For all $k \geq 3$ and $n \geq 2k$,*

$$rb(\mathcal{M}_n, kK_2) \leq n + 2k - 4 + 2\binom{2k-2}{2}.$$

Proof. Our proof is by induction on k. The cases for $k = 2,3$ will be verified in the next section and here we omit the checking details for these cases. Let M_n be any maximal outplanar graph of order n. For a given $(n+2k-4+2\binom{2k-2}{2})$-edge-coloring of M_n, let $G \subset M_n$ be a rainbow spanning subgraph with $|E(G)| = n + 2k - 4 + 2\binom{2k-2}{2}$. That is to say that G contains one and only one edge of each color. Since

$$n + 2k - 4 + 2\binom{2k-2}{2} > n + 2(k-1) - 4 + 2\binom{2k-4}{2},$$

by the induction assumption, the graph G contains a rainbow $(k-1)K_2$. Suppose that M_n does not contain any rainbow kK_2. Let

$$M = \{e_i = u_iw_i | i = 1, 2, \dots, k-1\}$$

be such a rainbow $(k-1)K_2$, and let $H = G[V(M)]$. Since $H \subset M_n$ and M_n be a maximal outerplanar graph, we have $|E(H)| \leq 2(2k-2) - 3 = 4k - 7$. Set $R = V(M_n) \setminus V(H)$ and it is clear that $|R| = n - 2k + 2$.

First observe that $E(G[R]) = \emptyset$, otherwise we have a rainbow kK_2 in G. Next we will estimate the number of $E_G(V(H), R)$, where $E_G(V(H), R)$ denotes the set of edges of G with an end in $V(H)$ and R, respectively. Let

$$d_1 \leq d_2 \leq \cdots \leq d_{n-2k+2}$$

be the degree sequence of all vertices of R in G. Then

$$\sum_{i=1}^{n-2k+2} d_i = |E_G(V(H), R)|.$$

If $d_i \leq 1$ for all $1 \leq i \leq n - 2k + 2$, then

$$\begin{aligned} |E(G)| &= |E(H)| + |E_G(V(H), R)| \\ &\leq (4k-7) + (n - 2k + 2) \\ &\leq n + 2k - 5, \end{aligned}$$

a contradiction to $|E(G)| = n + 2k - 4 + 2\binom{2k-2}{2}$.