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Note

A variant of Nim

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ABSTRACT

We discuss a version of Nim in which players are allowed to use a move from the traditional form of Nim or to split a pile after adding some predetermined number q of coins to the pile. When q is odd or negative, the play proceeds as in regular Nim. For q even and non-negative, we find three patterns: $q = 0$, $q = 2$ and $q \geqslant 4$.

1. Introduction and definitions

The winning strategy for the game of Nim was determined in 1902 by Bouton [2]. Since that seminal paper, there have been more than 100 scholarly articles on Nim and variants of Nim, and on the use of Nim and Sprague–Grundy numbers to solve other combinatorial games. For a general introduction to combinatorial games, see [1].

A combinatorial 2-person game is a game of complete information. Two players alternate moves. Both players are aware of the game position at any time, and there is no element of chance. The variant we consider is a normal game, meaning that the player who makes the last move wins.

In this version of Nim, which we call Split-q-Nim, two players begin with several piles of coins and then alternate moves, with the player who makes the last legal move declared the winner. The moves are of two types:

(*i*) A player can remove any non-zero number of coins from any pile; or

(*ii*) A player can split a pile of n coins by first adding q coins to the pile, $q \in \mathbb{Z}$, and then splitting the pile into two piles of heights at most $n - 1$.

By lexicographically ordering the possible game positions, it is obvious that no game position can be repeated. Also, from each position, there are only a finite number of game positions that may result from a single move.

We use $\mathbb{N} = \{a \in \mathbb{Z} : a \geqslant 0\}$ for the set of non-negative integers. For a set S which does not contain $\mathbb{N}$, we define $\text{mex}(S) = \min\{a \in \mathbb{N} : a \notin S\}$. For any position P, let $M(P)$ denote the positions that can be reached from P in a single move and let $\text{Gr}(P)$ denote the *Grundy number* of P, where $\text{Gr}(P) = \text{mex}(\{\text{Gr}(Q) : Q \in M(P)\})$. As in other combinatorial games, we can determine a winning strategy in our variant by calculating the Grundy number of each pile of coins, and playing the usual version of Nim on the Grundy numbers. Hence, it will suffice to calculate the Grundy number of a single pile of n coins, for all possible n.

We shall see in Section 3 that when q is odd or when q is negative, the Grundy number of a pile of height n is simply n, and a player has a winning strategy using only rule (*i*), if there is any winning strategy. Games with other values of q are more interesting.

In Section 2, we consider the case $q = 2$, where the Grundy number of a pile of height n depends on the number of ones in the base two representation of n. In Section 3, in addition to the cases in which q is odd or negative, we also consider the case $q = 0$, where the Grundy number depends on the last two binary digits of n, and the Grundy number of an even integer

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n can be n or $n-1$. In Section 4, we consider q even, $q > 0$, where the pattern depends on the last few binary digits of n, and the Grundy number of an even integer n can be n or $n+1$.

Let $[n_1, n_2, \ldots, n_k]$ represent the position with the jth pile having n_j coins, $n_j \geqslant 0$. We would like to determine the Grundy number $\mathrm{Gr}([n])$, $n \geqslant 0$. It seems convenient to write $\langle n_1, n_2, \ldots, n_k\rangle = \mathrm{Gr}([n_1, n_2, \ldots, n_k])$, saving a few symbols in some strings of equalities. We note that $\langle 0\rangle = 0$, for any q.

For non-negative integers a and b, we use $a \oplus b$ for the bitwise addition of a and b (also called Nimber addition), and $a \otimes b$ for the bitwise multiplication of a and b. Note that $a + b = (a \oplus b) + 2(a \otimes b) \geqslant a \oplus b$. Also, $a + b \equiv (a \oplus b) \pmod 2$. We shall use the commutativity and associativity of bitwise addition, as well as the cancellation law: $a \oplus b = a \oplus c \Rightarrow a \oplus (a \oplus b) = a \oplus (a \oplus c) \Rightarrow (a \oplus a) \oplus b = (a \oplus a) \oplus c \Rightarrow 0 \oplus b = 0 \oplus c \Rightarrow b = c$.

For a positive integer n, let $f(n)$ denote the number of non-zero bits in the binary representation of $2\left\lfloor \frac{n}{2} \right\rfloor$. Thus, if n is even, then $f(n)$ is the number of non-zero bits in the binary representation of n, and if n is odd, then $f(n)$ is the number of non-zero bits in the binary representation of $n \oplus 1 = n - 1$, and let $g(n) = 1$ if $f(n)$ is even and $g(n) = 0$ if $f(n)$ is odd. Note that $g(2k+1) = g(2k)$.

A *bit partition* of a positive integer m, with $f(m) \geqslant 2$, is an ordered pair of positive integers $[a, b]$ such that $a + b = a \oplus b = m$, $a \otimes b = 0$, $f(a+b) = f(a) + f(b)$, $f(a) \geqslant 1$, and $f(b) \geqslant 1$. It is obvious that when $f(m) \geqslant 2$, the positive integer m has a bit partition.

We say that k is *q-compliant* if $\{\mathrm{Gr}(n) : n \in \mathbb{Z},\ 0 \leqslant n < k\} = \{n \in \mathbb{Z} : 0 \leqslant n < k\}$ for the game Split-q-Nim. We say that $k = 2m$ is *super- q-compliant* if $\{\mathrm{Gr}(2j), \mathrm{Gr}(2j+1)\} = \{2j, 2j+1\}$ for the game Split-q-Nim, for every integer j, $0 \leqslant j < m$. Obviously, if k is super-q-compliant, then k is q-compliant. Also, if k is q -compliant, then $\mathrm{Gr}(k) \geqslant k$, for Split-$q$-Nim.

2. Split-2-Nim

The initial version we discuss is Split-2-Nim. The calculation of the Grundy numbers for $q = 2$ appears to be somewhat different than the calculation of the Grundy numbers for other values of q.

Observation 1. For $q = 2$, it is trivial to verify that $\langle k\rangle = k$ for $k \in \{0, 1, 2, 3, 4, 5\}$. For general q, $q > 0$, $\langle k\rangle = k$ for $0 \leqslant k \leqslant q+2$.

Proof. If $0 \leqslant k \leqslant q+1$, any split $[a, b]$, with $a + b = k + q$, has $\max\{a, b\} \geqslant k$. Hence, no split is possible. When $k = q+2$, the only possible split is $[q+1, q+1]$ and $\langle q+1, q+1\rangle = 0$. For the special case $q = 2$, we note that the only split for $k = 5$ is $[3, 4]$ and $\langle 3, 4\rangle = \langle 3\rangle \oplus \langle 4\rangle = 3 \oplus 4 = 7 \neq 5$. In all these cases, k cannot reach a position Q with $\mathrm{Gr}(Q) = k$ and, inductively, $\mathrm{Gr}(k) = k$. ■

Lemma 2. *Let $k = 2m$, $m \in \mathbb{N}$, $m \geqslant 3$. Suppose that $Gr(n) = n \oplus g(n)$, for $1 \leqslant n < k$. Then, $Gr(k) = k \oplus g(k)$.*

Proof. From the hypotheses, $\mathrm{Gr}(n) = n \oplus g(n)$, for $1 \leqslant n < k$, and, thus, $\mathrm{Gr}(n) \in \{n, n \oplus 1\}$ and k is super-2-compliant.

Note that for any $n = 2t < k$, if $\mathrm{Gr}(n) = n \oplus g(n)$, then $\mathrm{Gr}(n+1) = (n+1) \oplus g(n+1) = (n+1) \oplus g(n)$, and $\{\mathrm{Gr}(n), \mathrm{Gr}(n+1)\} = \{n, n+1\}$. Thus, $\{\mathrm{Gr}(n) : n \in \mathbb{Z},\ 0 \leqslant n < k\} = \{n \in \mathbb{Z} : 0 \leqslant n < k\}$.

Since $\langle a\rangle \oplus \langle b\rangle \in (a \oplus b, a \oplus b \oplus 1)$, $\langle a\rangle \oplus \langle b\rangle \leqslant (a \oplus b) + 1 = a + b - 2(a \otimes b) + 1$. If $a \otimes b \geqslant 2$, then $\langle a\rangle \oplus \langle b\rangle \leqslant a + b - 3 = 2m - 1$.

If $a \otimes b = 1$, then a and b are odd, and $[a-1, b-1]$ is a bit partition of $2m$. If $g(a-1) = g(b-1) = 1$, then $g(2m) = 1$ and $\langle a\rangle \oplus \langle b\rangle = (a \oplus 1) \oplus (b \oplus 1) = a \oplus b = 2m$. If $g(a-1) = g(b-1) = 0$, then $g(2m) = 1$ and $\langle a\rangle \oplus \langle b\rangle = a \oplus b = 2m$. If $g(a-1) \neq g(b-1)$, then $g(2m) = 0$ and $\langle a\rangle \oplus \langle b\rangle = a \oplus b \oplus 1 = 2m \oplus 1 = 2m+1$.

If $a \otimes b = 0$, then $[a, b]$ is a bit partition of $2m+2$, and both a and b are even. Then, $\langle a\rangle \oplus \langle b\rangle \in \{a \oplus b, a \oplus b \oplus 1\} = \{2m+2, 2m+3\}$.

In summary, for $a + b = 2m$, with $a \geqslant 3$ and $b \geqslant 3$:

(*i*) if $g(2m) = 0$, then $\langle a\rangle \oplus \langle b\rangle \in \{0, 1, \ldots, 2m-1, 2m+1, 2m+2, 2m+3\}$, and

(*ii*) if $g(2m) = 1$, then $\langle a\rangle \oplus \langle b\rangle \in \{0, 1, \ldots, 2m-1, 2m, 2m+2, 2m+3\}$.

Combining these results with $\{\mathrm{Gr}(n) : n \in \mathbb{Z},\ 0 \leqslant n < 2m\} = \{n \in \mathbb{Z} : 0 \leqslant n < 2m\}$, we have established that $\mathrm{Gr}(2m) = 2m \oplus g(2m)$. ■

To handle the odd cases, it is convenient to prove a more general lemma.

Lemma 3. *Suppose that $2m$ is super-q-compliant, for $q = 2s > 0$, $s \in \mathbb{N}$, and suppose that $\mathrm{Gr}(2m) = (2m) \oplus t \in \{2m, 2m+1\}$, $t \in \{0, 1\}$. Then, $\mathrm{Gr}(2m+1) = (2m+1) \oplus t \in \{2m, 2m+1\}$.*

Proof. From the hypotheses, $\{\mathrm{Gr}(n) : n \in \mathbb{Z},\ 0 \leqslant n < 2m+1\} = \{n \in \mathbb{Z} : 0 \leqslant n < 2m\} \cup \{\mathrm{Gr}(2m)\}$. Thus, from position $[2m+1]$, we can reach positions with any of the numbers in the above set by removing coins from the pile. Consider a move to $[a, b]$, with $a \geqslant q+1$ and $b \geqslant q+1$, if such a move is possible. Without loss of generality, a is odd, and $g(a-1) = g(a)$. Thus, $a = q+1$ or $[a-1, b]$ is a possible move from $[2m]$. Note that $\langle d\rangle = d$ for $0 \leqslant d \leqslant q+1$.

If $a = q+1$, then $b = 2m$. Since q is even, $(q+1) \oplus (2m) = q \oplus (2m+1)$ and, hence, $\langle a\rangle \oplus \langle b\rangle = (q+1) \oplus ((2m) \oplus t) = ((q+1) \oplus (2m)) \oplus t = (q \oplus (2m+1)) \oplus t = q \oplus ((2m+1) \oplus t)$. Since $q \neq 0$, $\langle a\rangle \oplus \langle b\rangle = q \oplus ((2m+1) \oplus t) \neq (2m+1) \oplus t$.