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Partial geometric difference sets and partial geometric difference families[☆]

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ABSTRACT

The concept of a partial geometric difference set (or $1\frac{1}{2}$-difference set) was introduced by Olmez in 2014. Recently, Nowak, Olmez and Song introduced the notion of a partial geometric difference family, which generalizes both the classical difference family and the partial geometric difference set. It was shown that partial geometric difference sets and partial difference families give rise to partial geometric designs. In this paper, a number of new infinite families of partial geometric difference sets and partial geometric difference families are constructed. From these partial geometric difference sets and difference families, we generate a list of infinite families of partial geometric designs.

1. Introduction

A *finite incidence structure* consists of a finite set $\mathcal{P}$ of *points*, a finite set $\mathcal{B}$ of *blocks* and an *incidence relation* between points and blocks. An incident point–block pair is called a *flag* and a nonincident point–block pair is called an *antiflag*. Let v, b, k and r be positive integers such that $v > k > 2$. A (v, b, k, r)-*tactical configuration* is a finite incidence structure $(\mathcal{P}, \mathcal{B})$ with $|\mathcal{P}| = v$, $|\mathcal{B}| = b$ such that each block contains k points and each point belongs to r blocks. A *partial geometric design* with parameters $(v, b, k, r; \alpha, \beta)$ is a (v, b, k, r)-tactical configuration $(\mathcal{P}, \mathcal{B})$ satisfying the "partial geometric" property:

For every point $x \in \mathcal{P}$ and every block $B \in \mathcal{B}$, the number of flags (y, C) such that $y \in B$ and $x \in C$ is β if $x \in B$, and is α if $x \notin B$.

Examples of partial geometric designs include classical 2-designs, transversal designs, and partial geometries. The study of partial geometric designs provides links among algebra, geometry, graphs, and design theory.

The notion of a partial geometric design was initially introduced by Bose, Shrikhande, and Singhi [5] in 1976. Subsequently, Bose, Bridges and Shrikhande [2–4] investigated algebraic and combinatorial properties of partial geometric designs. In the process of study of $t\frac{1}{2}$-designs in 1980, Neumaier [12] called partial geometric designs $1\frac{1}{2}$-*designs* as a subclass. In 2005 van Dam and Spence [19] studied partial geometric designs as a type of combinatorial designs whose incidence matrix has two distinct singular values. Chai et al. [7] constructed partial geometric designs from symplectic geometry over finite fields.

In 2013, Olmez [16] introduced the concept of a partial geometric difference set, under the name of "$1\frac{1}{2}$-difference set", as a difference set version of partial geometric design and he demonstrated that we can obtain symmetric partial geometric designs from partial geometric difference sets in precisely the same manner that we obtain symmetric 2-designs from ordinary difference sets. Recently, Nowak et al. [15] introduced the notion of a partial geometric difference family, which generalizes both the classical difference family and the partial geometric difference set. It was shown that a partial geometric difference family also gives rise to a partial geometric design. After that, Michel [11] constructed several new

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infinite classes of partial geometric difference sets and partial geometric difference families, mostly by similar approaches of Nowak et al. [14,15], but also from planar functions. Very recently, Davis and Olmez [8] used the framework of extended building sets to find infinite families of partial geometric difference sets in abelian groups.

This paper focuses on general constructions for partial geometric difference sets and partial geometric difference families. The rest is organized as follows. Section 2 recalls some preliminary notation and terminology that will be used throughout the paper. In Section 3, we display a few general constructions of partial geometric difference sets in the direct product of two groups. Our constructions unify and generalize a few known results of Michel [11], Olmez [17], and Spence [18]. With applications we get a number of new families of partial geometric difference sets. In Section 4 we produce several new families of partial geometric difference families. Finally, in Section 5, we include some remarks on the partial geometric designs obtained from our previous constructions.

2. Preliminaries

Let S be a subset of a finite group G. We define a multiset $\triangle S = [st^{-1} : s, t \in S]$. Given a family $\mathcal{F} = \{S_1, S_2, \dots, S_l\}$ of l distinct k-element subsets of G, for each element $z \in G$, we define $\delta_j(z)$ to be the multiplicity of z in $\triangle S_j$. Let $\delta(z)$ denote the sum of $\delta_j(z)$ over all j; that is,

$$\delta(z) = \sum_{j=1}^{l} \delta_j(z).$$

Let v, k and l be positive integers with $v > k > 2$. Let G be a group of order v. A family $\mathcal{F} = \{S_1, S_2, \dots, S_l\}$ of distinct k-element subsets of G is called a *partial geometric difference family* (PGDF, Nowak et al. [15]) in G with parameters $(v, k, l; \alpha, \beta)$ if there exist constants α and β such that, for each $x \in G$ and every $i \in \{1, 2, \dots, l\}$,

$$\sum_{y \in S_i} \delta\left(xy^{-1}\right) = \begin{cases} \alpha & x \notin S_i, \\ \beta & x \in S_i. \end{cases} \tag{2.1}$$

In particular, when $l = 1$, the partial geometric difference family $\mathcal{F} = \{S_1\}$ is called a *partial geometric difference set* (PGDS for short) with parameters $(v, k; \alpha, \beta)$.

For any $g \in G$ and $S \in \mathcal{F}$, we define the *translate* of S by $Sg = \{sg : s \in S\}$, and define the *development* of $\mathcal{F}$ by $Dev(\mathcal{F}) = [Sg : S \in \mathcal{F}, g \in G]$.

Theorem 2.1 *([15, Theorem 3]). Let $\mathcal{F} = \{S_1, \dots, S_l\}$ be a family of distinct k-subsets of a group G of order v. If $\mathcal{F}$ is a partial geometric difference family with parameters $(v, k, l; \alpha, \beta)$, then $(G, Dev(\mathcal{F}))$ is a partial geometric design with parameters $(v, vl, k, kl; \alpha, \beta)$.*

It is standard and convenient to use the language of group rings to study difference sets. Let G be a finite group and let $\mathbb{Z}[G]$ be the group ring of G. By the definition, $\mathbb{Z}[G]$ is the ring of formal sums

$$\mathbb{Z}[G] = \left\{\sum_{g \in G} a_g g : a_g \in \mathbb{Z}\right\}.$$

The ring $\mathbb{Z}[G]$ has the operation of addition and multiplication given by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g,$$
$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} a_g b_h gh.$$

For a nonempty set $S \subseteq G$, we use the same S to denote the group ring element $\sum_{s \in S} s$, and define $S^{-1} = \sum_{s \in S} s^{-1}$.

Theorem 2.2 *([16, Lemma 2.8]). Let G be a group of order v and S be a subset of G of size k. Then, S is a partial geometric difference set with parameters $(v, k; \alpha, \beta)$ in G if and only if the group ring equation holds:*

$$SS^{-1}S = nS + \alpha G \tag{2.2}$$

where $n = \beta - \alpha$.

Remark 2.3. In the definition of a partial geometric difference set, the value of β defined by Olmez [16, Definition 2.2] is $2k - 1$ less than the β defined in our (2.1). So it is not difficult to have that the value of n in (2.2) should be $\beta - \alpha$ instead of $2k - 1 + \beta - \alpha$.

For the rest of the paper the parameter n will denote the number $\beta - \alpha$ for a given partial geometric difference set with parameters $(v, k; \alpha, \beta)$.