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An improved upper bound for the order of mixed graphs

C. Dalfó [a,1], M.A. Fiol [b], N. López [c]

[a] Dept. de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain
[b] Dept. de Matemàtiques, Barcelona Graduate School of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Spain
[c] Dept. de Matemàtica, Universitat de Lleida, Lleida, Spain

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ABSTRACT

A mixed graph G can contain both (undirected) edges and arcs (directed edges). Here we derive an improved Moore-like bound for the maximum number of vertices of a mixed graph with diameter at least three. Moreover, a complete enumeration of all optimal $(1, 1)$-regular mixed graphs with diameter three is presented, so proving that, in general, the proposed bound cannot be improved.

1. Introduction

A *mixed* (or *partially directed*) graph $G = (V, E, A)$ consists of a set V of vertices, a set E of edges, or unordered pairs of vertices, and a set A of arcs, or ordered pairs of vertices. Thus, G can also be seen as a digraph having *digons*, or pairs of opposite arcs between some pairs of vertices. If there is an edge between vertices $u, v \in V$, we denote it by $u \sim v$, whereas if there is an arc from u to v, we write $u \to v$. We denote by $r(u)$ the *undirected degree* of u, or the number of edges incident to u. Moreover, the *out-degree* [respectively, *in-degree*] of u, denoted by $z^+(u)$ [respectively, $z^-(u)$], is the number of arcs emanating from [respectively, to] u. If $z^+(u) = z^-(u) = z$ and $r(u) = r$, for all $u \in V$, then G is said to be *totally regular* of degrees (r, z), with $r + z = d$ (or simply (r, z)-*regular*). The length of a shortest path from u to v is the *distance* from u to v, and it is denoted by $\mathrm{dist}(u, v)$. Note that $\mathrm{dist}(u, v)$ may be different from $\mathrm{dist}(v, u)$ when the shortest paths between u and v involve arcs. The maximum distance between any pair of vertices is the *diameter* k of G. Given $i \le k$, the set of vertices at distance i from vertex u is denoted by $G_i(u)$.

As in the case of (undirected) graphs and digraphs, the degree/diameter problem for mixed graphs calls for finding the largest possible number of vertices $N(r, z, k)$ in a mixed graph with maximum undirected degree r, maximum directed outdegree z, and diameter k. A bound for $N(r, z, k)$ is called a Moore(-like) bound. It is obtained by counting the number of vertices of a *Moore tree* $MT(u)$ rooted at a given vertex u, with depth equal to the diameter k, and assuming that for any vertex v there exists a unique shortest path of length at most k (with the usual meaning when we see G as a digraph) from u to v. The number of vertices in $MT(u)$, which is denoted by $M(r, z, k)$, was given by Buset, Amiri, Erskine, Miller, and Pérez-Rosés [1], and it is the following:

$$M(r, z, k) = A\frac{u_1^{k+1} - 1}{u_1 - 1} + B\frac{u_2^{k+1} - 1}{u_2 - 1}, \tag{1}$$

E-mail addresses: cristina.dalfo@upc.edu (C. Dalfó), miguel.angel.fiol@upc.edu (M.A. Fiol), nlopez@matematica.udl.es (N. López).

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Table 1
Moore bounds according to (1).

d	k				
	1	2	3	4	5
1	**2**	$z+\mathbf{2}$	$2z+\mathbf{2}$	$z^2+2z+\mathbf{2}$	$2z^2+2z+\mathbf{2}$
2	**3**	$z+\mathbf{5}$	$4z+\mathbf{7}$	$z^2+9z+\mathbf{9}$	$5z^2+16z+\mathbf{11}$
3	**4**	$z+\mathbf{10}$	$6z+\mathbf{22}$	$z^2+22z+\mathbf{46}$	$8z^2+66z+\mathbf{94}$
4	**5**	$z+\mathbf{17}$	$8z+\mathbf{53}$	$z^2+41z+\mathbf{161}$	$11z^2+176z+\mathbf{485}$
5	**6**	$z+\mathbf{26}$	$10z+\mathbf{106}$	$z^2+66z+\mathbf{426}$	$14z^2+370z+\mathbf{1706}$

where

$$v=(z+r)^2+2(z-r)+1,$$
$$u_1=\frac{z+r-1-\sqrt{v}}{2},\qquad u_2=\frac{z+r-1+\sqrt{v}}{2},$$
$$A=\frac{\sqrt{v}-(z+r+1)}{2\sqrt{v}},\qquad B=\frac{\sqrt{v}+(z+r+1)}{2\sqrt{v}}.$$

This bound applies when G is totally regular with degrees (r,z). Moreover, if we bound the total degree $d=r+z$, the largest number is always obtained when $r=0$ and $z=d$. That is, when the mixed graph has no (undirected) edges. In Table 1 we show the values of (1) when $r=d-z$, with $0\le z\le d$, for different values of d and diameter k. In particular, when $z=0$, the bound corresponds to the Moore bound for graphs (numbers in bold).

2. A new upper bound

An alternative approach for computing the bound given by (1) is the following (see also [2]). Let G be a (r,z)-regular mixed graph with $d=r+z$. Given a vertex v and for $i=0,1,\dots,k$, let $N_i=R_i+Z_i$ be the maximum possible number of vertices at distance i from v. Here, R_i is the number of vertices that, in the corresponding tree rooted at v, are adjacent by an edge to their parents; and Z_i is the number of vertices that are adjacent by an arc from their parents. Then,

$$N_i=R_i+Z_i=R_{i-1}((r-1)+z)+Z_{i-1}(r+z). \tag{2}$$

That is,

$$R_i=R_{i-1}(r-1)+Z_{i-1}r, \tag{3}$$
$$Z_i=R_{i-1}z+Z_{i-1}z, \tag{4}$$

or, in matrix form,

$$\begin{pmatrix}R_i\\Z_i\end{pmatrix}=\begin{pmatrix}r-1 & r\\ z & z\end{pmatrix}\begin{pmatrix}R_{i-1}\\Z_{i-1}\end{pmatrix}=\cdots=\boldsymbol{M}^i\begin{pmatrix}R_0\\Z_0\end{pmatrix}=\boldsymbol{M}^i\begin{pmatrix}0\\1\end{pmatrix},$$

where $\boldsymbol{M}=\begin{pmatrix}r-1 & r\\ z & z\end{pmatrix}$ and, by convenience, $R_0=0$ and $Z_0=1$. Therefore,

$$N_i=R_i+Z_i=\begin{pmatrix}1 & 1\end{pmatrix}\boldsymbol{M}^i\begin{pmatrix}0\\1\end{pmatrix}.$$

Consequently, after summing a geometric matrix progression, the order of $MT(u)$ turns out to be

$$M(r,z,k)=\sum_{i=0}^{k}N_i=\frac{1}{r+2z-2}\begin{pmatrix}1 & 1\end{pmatrix}(\boldsymbol{M}^{k+1}-\boldsymbol{I})\begin{pmatrix}r\\z\end{pmatrix}, \tag{5}$$

with $r+2z\neq 2$, that is, except for the cases $(r,z)=(0,1)$ and $(r,z)=(2,0)$, which correspond to a directed and undirected cycle, respectively.

Alternatively, note that N_i satisfies an easy linear recurrence formula (see again Buset, El Amiri, Erskine, Miller, and Pérez-Rosés [1]). Indeed, from (2) and (4) we have that $Z_i=z(N_{i-1}-Z_{i-1})+zZ_{i-1}=zN_{i-1}$ and, hence,

$$\begin{aligned}N_i&=(r+z)N_{i-1}-R_{i-1}=(r+z)N_{i-1}-(N_{i-1}-Z_{i-1})\\&=(r+z-1)N_{i-1}+zN_{i-2},\qquad i=2,3,\dots\end{aligned} \tag{6}$$

with initial values $N_0=1$ and $N_1=r+z$.

In this context, Nguyen, Miller, and Gimbert [3] showed that the bound in (1) is not attained for diameter $k\ge 3$ and, hence, that *mixed Moore graphs* do not exist in general. More precisely, they proved that there exists a pair of vertices u,v such that