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On 3-stable number conditions in n-connected claw-free graphs

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ABSTRACT

For a subgraph X of G, let $\alpha_G^3(X)$ be the maximum number of vertices of X that are pairwise distance at least three in G. In this paper, we prove three theorems. Let n be a positive integer, and let H be a subgraph of an n-connected claw-free graph G. We prove that if $n \geq 2$, then either H can be covered by a cycle in G, or there exists a cycle C in G such that $\alpha_G^3(H - V(C)) \leq \alpha_G^3(H) - n$. This result generalizes the result of Broersma and Lu that G has a cycle covering all the vertices of H if $\alpha_G^3(H) \leq n$. We also prove that if $n \geq 1$, then either H can be covered by a path in G, or there exists a path P in G such that $\alpha_G^3(H - V(P)) \leq \alpha_G^3(H) - n - 1$. By using the second result, we prove the third result. For a tree T, a vertex of T with degree one is called a leaf of T. For an integer $k \geq 2$, a tree which has at most k leaves is called a k-ended tree. We prove that if $\alpha_G^3(H) \leq n + k - 1$, then G has a k-ended tree covering all the vertices of H. This result gives a positive answer to the conjecture proposed by Kano et al. (2012).

1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. Let G be a graph. Let $V(G)$ and $E(G)$ be the set of vertices of G and the set of edges of G, respectively. Let $\alpha(G)$ be the independence number of G. We sometimes write $|G|$ for the order of G. For a vertex v of G, let $N_G(v)$ be the neighborhood of v in G. For two distinct vertices x and y of G, let $d_G(x, y)$ be the distance between x and y in G. We call a vertex set of G a *3-stable set* if the distance between each pair of distinct vertices of it is at least 3. For a subgraph (or subset) H of G (or $V(G)$), let $\alpha_G^3(H) = \max\{|S| : S \subseteq V(H)$ (or $S \subseteq H$) is a 3-stable set of $G\}$. A graph G is *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$.

Chvátal and Erdős gave an independence number condition for a graph to have a hamiltonian cycle as follows.

Theorem 1 (*Chvátal and Erdős [3]*). *Let $n \geq 2$ be a positive integer, and let G be an n-connected graph. If $\alpha(G) \leq n$, then G has a hamiltonian cycle.*

A hamiltonian cycle is a cycle containing all the vertices of a graph. In this sense, Fournier considers a cycle containing specified vertices as a generalization of a hamiltonian cycle.

Theorem 2 (*Fournier [4]*). *Let $n \geq 2$ be an integer, and let H be a subgraph of an n-connected graph G. If $\alpha(H) \leq n$, then G has a cycle covering all the vertices of H.*

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Kouider generalized above two theorems as follows.

Theorem 3 (*Kouider [6]*). *Let $n \geq 2$ be an integer, and let H be a subgraph of an n-connected graph G. Then either H can be covered by a cycle in G, or there exists a cycle C in G such that $\alpha(H - V(C)) \leq \alpha(H) - n$.*

It seems that the situation is quite different if we consider claw-free graphs. But, by using $\alpha^3_G(*)$ instead of $\alpha(*)$, Broersma and Lu obtained the following analog of Theorem 1 for claw-free graphs.

Theorem 4 (*Broersma and Lu [2]*). *Let $n \geq 2$ be an integer, and let H be a subgraph of an n-connected claw-free graph G. If $\alpha^3_G(H) \leq n$, then G has a cycle covering all the vertices of H.*

In this paper, we show that the analog of Theorem 3 for claw-free graphs also holds. We prove the following theorem.

Theorem 5. *Let $n \geq 2$ be an integer, and let H be a subgraph of an n-connected claw-free graph G. Then either H can be covered by a cycle in G, or there exists a cycle C in G such that $\alpha^3_G(H - V(C)) \leq \alpha^3_G(H) - n$.*

By modifying the proof of Theorem 5, we also prove the following theorem.

Theorem 6. *Let n be a positive integer, and let H be a subgraph of an n-connected claw-free graph G. Then either H can be covered by a path in G, or there exists a path P in G such that $\alpha^3_G(H - V(P)) \leq \alpha^3_G(H) - n - 1$.*

By using Theorem 6, we prove the third main theorem which gives a positive answer to the conjecture proposed by Kano et al. in [5]. Let G be a graph. A vertex of G with degree one is called a *leaf*, and the set of leaves of G is denoted by *Leaf*(G). A tree having at most k leaves is called a *k-ended tree*. For a graph G, $\sigma_k(G)$ denotes the minimum degree sum of k independent vertices of G if $\alpha(G) \geq k$. Otherwise, we set $\sigma_k(G) = \infty$. The following theorem gives a degree sum condition for a graph to have a spanning k-ended tree.

Theorem 7 (*Broersma and Tuinstra [1]*). *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

Kano et al. show that the condition becomes much weaker if the graph is claw-free.

Theorem 8 (*Kano, Kyaw, Matsuda, Ozeki, Saito and Yamashita [5]*). *Let $k \geq 2$ be an integer, and let G be a connected claw-free graph. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning k-ended tree.*

In [5], they also proposed the following conjecture. Note that in [7], Ryjáček obtained that every 7-connected claw-free graph is hamiltonian.

Conjecture 9. *Let k and n be integers such that $k \geq 2$ and $2 \leq n \leq 6$, and let G be an n-connected claw-free graph. If $\sigma_{n+k}(G) \geq |G| - n - k + 1$, then G has a spanning k-ended tree.*

We obtain the following result which implies Theorem 8 and Conjecture 9.

Theorem 10. *Let $k \geq 2$ and $n \geq 1$ be integers, and let H be a subgraph of an n-connected claw-free graph G. If $\alpha^3_G(H) \leq n+k-1$, then G has a k-ended tree covering all the vertices of H.*

Proof of Theorem 8 and Conjecture 9. If there exists a 3-stable set S of G such that $|S| = n+k$, then $\sigma_{n+k}(G) \leq \sum_{v \in S}|N_G(v)| \leq |G| - (n+k) < |G| - n - k + 1$. Therefore, $\sigma_{n+k}(G) \geq |G| - n - k + 1$ implies $\alpha^3_G(G) \leq n + k - 1$. Hence, Theorem 10 implies Theorem 8 and Conjecture 9. □

2. Proof of Theorem 5

In order to prove Theorem 5, we first introduce some notation and facts. Let C be an oriented cycle (or a path) and $x, y \in V(C)$. We denote the successor and the predecessor of x on C by x^+ and x^-, respectively. And we denote by $C[x, y]$ a path from x to y along the orientation of C. We denote $C[x, y] - \{x, y\}$, $C[x, y] - \{x\}$, $C[x, y] - \{y\}$ by $C(x, y)$, $C(x, y]$ and $C[x, y)$, respectively. Let G be a graph, and H be a subgraph of G. For $x \in V(G) - V(H)$, we call *x is insertible into H* if there exist $u, u' \in V(H)$ such that $uu' \in E(H)$ and $xu, xu' \in E(G)$. We can easily see that the following two facts hold by the definition of $\alpha^3_G(*)$.

Fact 1. *Let H and H' be subgraphs of a graph G such that $V(H') \subseteq V(H)$. Then $\alpha^3_G(H') \leq \alpha^3_G(H)$.*

Fact 2. *Let H and H' be subgraphs of a graph G such that $d_G(x, y) \geq 3$ for any $x \in V(H)$ and $y \in V(H')$. Then $\alpha^3_G(H \cup H') = \alpha^3_G(H) + \alpha^3_G(H')$.*