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Perspective

# A new sufficient condition for a toroidal graph to be 4-choosable

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A graph $G$ is $k$-choosable if $G$ can be properly colored whenever every vertex has a list of at least $k$ available colors. In this paper, we will proof that if every 5-cycle of toroidal graph $G$ is not adjacent simultaneously to 3-cycles and 4-cycles, then $G$ is 4-choosable. This improves a result shown in Xu and Wu (2017), which stated that if every 5-cycle of planar graph $G$ is not adjacent simultaneously to 3-cycles and 4-cycles, then $G$ is 4-choosable.



## 1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow from [2] for terminologies and notations not defined here.

A *proper coloring* of a graph $G$ is an assignment $c$ of integers to the vertices of $G$ such that $c(u) \neq c(v)$ for any two adjacent vertices $u, v$. For a given list assignment $L = \{L(v)|v \in V(G)\}$, a graph $G$ is *list L-colorable* if there exists a proper coloring $c$ of the vertices such that $c(v) \in L(v)$ for all $v \in V(G)$. If $G$ is list $L$-colorable for every list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is *k-choosable*. The list chromatic number, denoted by $\chi_l(G)$, is the least integer $k$ such that $G$ is $k$-choosable.

A graph $G$ is *toroidal* if it can be drawn on the torus so that the edges meet only at the vertices. For a graph $G$ embedded into the totus, we use $V(G), E(G), F(G)$ (or simple $V, E, F$ ) to denote its vertex set, edge set and face set, respectively. For a vertex $v \in V$, the *degree* of $v$, denoted by $d(v)$, is the number of edges incident with $v$ in $G$. For a face $f \in F$, the *degree* of $f$, denoted by $d(f)$, is the number of edges incident with $f$ in $G$ (a cut-edge is counted twice). A vertex $v \in V$ is called a *$k$-, $k^+$-, or $k^-$-vertex* if $d(v) = k, \geq k$, or $\leq k$, respectively. The notion of a $k$-, $k^+$-, or $k^-$-face is similarly defined. The *minimum degree* of $G$, $\min\{d(v)|v \in V\}$, is denoted by $\delta(G)$. We say that two cycles (or faces) are *adjacent* if they share at least one edge. Two cycles (or faces) are *normally adjacent* if they share exactly one edge. A *chord* of a cycle $C$ is an edge that connects two non-consecutive vertices of $C$. For convenience, we denote by $n_d(f)$ the number of $d$-vertices incident with the face $f$. Similarly, we can define $n_{d+}(f)$, $n_{d-}(f)$. A face $f$ is called *light* if $n_4(f) = d(f)$.

The concept of list-coloring was introduced by Vizing [10] and independently by Erdős et al. [6]. Thomassen [9] proved that every planar graph is 5-choosable, whereas Voigt [11] presented an example of a planar graph which is not 4-choosable. It is proved in [1] that every toroidal graph $G$ is 7-choosable, and $\chi_l(G) = 7$ if and only if $K_7 \subseteq G$. More recently, Cai et al. [3] proved that if $G$ is a toroidal graph with no cycles of a fixed length $k$, then (1) $\chi_l(G) \leq 4$ if $k \in 3, 4, 5$; (2) $\chi_l(G) \leq 5$ if $k = 6$; and (3) $\chi_l(G) \leq 6$ if $k = 7$, and moreover $\chi_l(G) = 6$ if and only if $K_6 \subseteq G$. Luo [8] proved that every toroidal graph without intersecting triangles is 4-choosable. Xu and Wu [12] proved that every 5-cycle of planar graph $G$ is not simultaneously

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adjacent to 3-cycles and 4-cycles, then $G$ is 4-choosable. In this paper, we generalize the result in [12] to the toroidal graph by showing the following theorem:

**Theorem 1.** *Let $G$ be a toroidal graph. If every 5-cycles of $G$ is not adjacent simultaneously to 3-cycles and 4-cycles, then $G$ is 4-choosable.*

## 2. Proof of Theorem 1

Arguing by contradiction, we assume that $G = (V, E)$ is a counterexample to Theorem 1 having the fewest vertices. Embedding $G$ into the torus, then $G$ has the following properties:

**P1** $\delta(G) \geq 4$ (see [7]).

**P2** $G$ does not contain any induced even cycle $C$ such that each vertex of $C$ is of degree 4 (see [5]).
Since every 5-cycle of $G$ is not adjacent simultaneously to 3-cycles and 4-cycles, P3 and P4 hold.

**P3** Any 5-cycle has no chord.

**P4** If two 3-faces are adjacent, then each other face adjacent to at least one of them is a $6^+$-face.
A $\theta$-*graph* is a one consisting of two 3-vertices and three pairwise internally disjoint paths between the two 3-vertices. Clearly, a $k$-cycle with one internal chord is a special $\theta$-graph. A $\theta$-*subgraph* of $G$ is an induced subgraph that is isomorphic to a $\theta$-graph. Furthermore, we use $S\theta$ to denote such a special $\theta$-subgraph of $G$ in which one of the ends of the internal chord is a $5^-$-vertex and all of the other vertices are 4-vertices in $G$.

**P5** $G$ contains no $S\theta$ (see [4]).

**Proof.** Assume to the contrary that $G$ has an $S\theta$. Let $H$ be an $S\theta$ of $G$ with the internal chord $e = x_0x$, where $x$ is a $5^-$-vertex. That is, $H = C + \{x_0x\}$. Let $L$ be any list assignment of $G$ with $|L(v)| \geq 4$ for all $v \in V(G)$. Deleting $H$ from $G$, we get a smaller graph $G'$. By the minimality of $G$, $G'$ is $L$-colorable. Let $\phi$ be an $L$-coloring of $G'$. We define a list assignment $L'(v) = L(v) \setminus \{\phi(v')|v'v \in E(G), v' \in V(G) \setminus V(H)\}$ for every $v \in V(H)$. By the definition of an $S\theta$, $|L'(x_0)| \geq 3$ and $|L'(v)| \geq 2$ for every $v \in V(H) \setminus \{x_0\}$. If, for every $v \in V(H) \setminus \{x_0\}$, we have $\{\alpha, \beta\} \subseteq L'(v)$. Then we can choose a color from $L'(x_0) \setminus \{\alpha, \beta\}$ to color $x_0$, and using $\alpha$ and $\beta$ color all other vertices of $H$ alternatively in a cyclic order. Next, suppose that there are two adjacent vertices on the path $P = C \setminus \{x_0\}$, say $u$ and $w$, such that $L'(u) \neq L'(w)$. Without loss of generality, assume that $u$ is closer to $x_0$ than $w$ on the cycle $C$. Now we can first choose a color from $L'(w) \setminus L'(u)$ to color $w$, then color all the remaining vertices of $C$ in a chosen cyclic order such that $u$ is colored finally, giving an $L$-coloring of $G$, a contradiction.

If a light 5-face $P = [v_1v_2 \cdots v_5]$ is adjacent to a 3-face $T = [v_1v_2u]$, then $u \notin V(P)$ by P3. We call $u$ a *source* of $P$ through $T$, and $P$ a *sink* of $u$ through $T$.

**Lemma 2.** (1) *Every light 4-face must be adjacent to a $6^+$-face.*
(2) *If a 5-cycle $P$ is adjacent to a 3-cycle $T$, then they are normally adjacent and $P \cup T$ is a $\theta$-subgraph.*
(3) *Let $u$ be a source of a light 5-face $P$. Then $d(u) \geq 5$.*
(4) *$G$ contains no 6-cycle $C$ such that each vertex of $C$ is of degree 4.*

**Proof.** (1) Let $f = [v_1v_2v_3v_4]$ be the 4-face with $d(v_i) = 4$ for $i = 1, 2, 3, 4$. Then $f$ must have a chord by P4, say $v_1v_3 \in E$. Hence, it is easy to check that each face adjacent to $f$ is a $6^+$-face, since every 5-cycle of $G$ is not adjacent simultaneously to 3-cycles and 4-cycles.

(2) It is clear that $P$ and $T$ are normally adjacent by P3. Next prove that $P \cup T$ is a $\theta$-subgraph. Let $P = v_1v_2v_3v_4v_5v_1$ be the 5-cycle, and $T = v_1v_2uv_1$ be the 3-cycle adjacent to $P$. Then $C = uv_2v_3v_4v_5v_1u$ is a 6-cycle of $G$. If $P \cup T$ is not a $\theta$-subgraph, then $C$ has one more chord other than $v_1v_2$. By P3, the second chord of $C$ must be $uv_3$ or $uv_4$. Hence, $G$ has a 5-cycle $v_1v_2v_3v_4v_5v_1$ adjacent to 3-cycle $v_1uv_2v_1$ and 4-cycle $v_1uv_3v_2v_1$ or $v_1v_5v_4uv_1$, a contradiction. So $P \cup T$ is a $\theta$-subgraph.

(3) Let $P = [v_1v_2v_3v_4v_5]$ be the light 5-face, and $T = [v_1v_2u]$ be the 3-face adjacent to $P$. By Lemma 2(2), $P \cup T$ is a $\theta$-subgraph of $G$, so $d(u) \geq 5$ by P5.

(4) Let $C = v_1v_2v_3v_4v_5v_6v_1$ be the 6-cycle of $G$ with $d(v_i) = 4$, where $i = 1, 2, \ldots, 6$. By P2 and P5, $C$ has at least two chords. If $C$ has a chord $v_1v_3$, then there exists a 5-cycle $C_1 = v_1v_3v_4v_5v_6v_1$ adjacent to 3-cycle $T = v_1v_2v_3v_1$. By Lemma 2(2) and the fact that $d(v_i) = 4$ for every $i = 1, 2, \ldots, 6$, $C_1 \cup T$ is an $S\theta$, a contradiction. Next, assume that $C$ has the chord $v_1v_4$. By the above discussion, we have $v_1v_3 \notin E$, $v_2v_4 \notin E$. That is, $v_1v_2v_3v_4v_1$ is an induced 4-cycle with $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 4$, a contradiction to P2.

By Euler's formula $|V| - |E| + |F| = 0$, we have

$$\sum_{v \in V}(d(v) - 4) + \sum_{f \in F}(d(f) - 4) = -4(|V| - |E| + |F|) = 0.$$