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Descendant sets in infinite, primitive highly arc transitive digraphs with prime power out-valency

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ABSTRACT

The descendant set desc(α) of a vertex α in a directed graph (digraph) is the subdigraph on the set of vertices reachable by a directed path from α. We study the structure of descendant sets Γ in an infinite, primitive, highly arc transitive digraph with out-valency p^k, where p is a prime and $k \geq 1$. It was already known that Γ is a tree when $k = 1$ and we show the same holds when $k = 2$. However, for $k \geq 3$ there are examples of infinite, primitive highly arc transitive digraphs of out-valency p^k whose descendant sets are not trees, for some prime p.

1. Introduction

Descendant sets in highly arc transitive digraphs of finite out-valency were studied by the Author in [1,2], following on from results obtained by Möller for locally finite, highly arc transitive digraphs in [8]. This work isolates a small number of quite simple properties (essentially H0, H1, H2 and H3 in Section 3) satisfied by such descendant sets and shows that these properties have rather strong structural consequences. In particular, the descendant set admits a non-trivial, finite-to-one homomorphism onto a tree. Digraphs having the given properties, but which are not trees are constructed in [2,8]. Moreover, (imprimitive) highly arc transitive digraphs having these as descendant sets are constructed in [2,4,6].

In particular, it is shown in [2] that if Γ is a digraph having the given properties with prime out-valency p then Γ is a tree. In this paper we study digraphs Γ with out-valency p^k, $k \geq 1$. Our main result is Proposition 3.2 which shows that if Γ is a digraph satisfying H0 to H3 then there are subsets $\sigma_1, \ldots, \sigma_p$ in the first layer Γ^1 of Γ such that if $i \neq j$ then the descendant sets desc(σ_i) and desc(σ_j) are disjoint. As a consequence we obtain that if Γ is a digraph satisfying H0 to H3 with out-valency p^2, then Γ is a tree (see Corollary 3.4). This result gives a positive answer to a question posed in [2]. On the other hand, in Section 4, we point out that for $k \geq 3$ there are digraphs with out-valency p^k having the given properties and which are not trees.

1.1. Notation and terminology

A digraph $(D; E(D))$ consists of a set D of vertices, and a set $E(D) \subseteq D \times D$ of ordered pairs of vertices, the (directed) edges. Our digraphs will have no loops and no multiple edges. Also, we generally exclude the case of null digraphs, where there are no edges. We will think of a subset X of the set D of vertices as a digraph in its own right by considering the full induced subdigraph on X (so $E(X) = E(D) \cap X^2$). Throughout this paper, 'subdigraph' will mean 'full induced subdigraph'. Thus henceforth, we will not usually distinguish notationally between a digraph and its vertex set. In particular, we will usually

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refer to the digraph $(D; E(D))$ simply as 'the digraph D'. Note that this is a different convention from the usual notation $D = (VD; ED)$. Furthermore, we will use notation such as '$\alpha \in D$' to indicate that α is a vertex of the digraph D.

We denote the automorphism group of the digraph D by $\mathrm{Aut}(D)$. We say that D is transitive (respectively, edge transitive) if this is transitive on D (respectively, $E(D)$). We say that D is *primitive* if $\mathrm{Aut}(D)$ is primitive on D, that is, there are no non-trivial $\mathrm{Aut}(D)$-invariant equivalence relations on D.

The *out-valency* of a vertex $\alpha \in D$ is the size of the set $\{u \in D : (\alpha, u) \in E(D)\}$ of out-neighbours of α; similarly, the *in-valency* of α is the size of the set $\{u \in D : (u, \alpha) \in E(D)\}$ of in-neighbours. Let $s \geq 0$ be an integer. An *s-arc* from u to v in D is a sequence $u_0u_1 \ldots u_s$ of $s + 1$ vertices such that $u_0 = u$, $u_s = v$ and $(u_i, u_{i+1}) \in ED$ for $0 \leq i < s$ and $u_{i-1} \neq u_{i+1}$ for $0 < i < s$. Usually our digraphs will be asymmetric, in which case this last condition is redundant. We denote by $D^s(u)$ the set of vertices of D which are reachable by an s-arc from u. The *descendant set* $D(u)$ (or $\mathrm{desc}(u)$) of u is $\bigcup_{s\geq 0} D^s(u)$. Similarly the set $\mathrm{anc}(u)$ of *ancestors* of u is the set of vertices of which u is a descendant, and $D^{-s}(u)$ denotes the set of vertices x in D for which there is an s-arc from x to u. In particular, fix $\alpha \in D$, and let $\Gamma = D(\alpha)$. If $\mathrm{Aut}(D)$ is transitive on the set of vertices of D, then $D(u) \cong \Gamma$ for all vertices u, and we shall speak of the digraph Γ as *the descendant set* of D.

We say that the digraph D is *highly arc transitive* if for each $s \geq 0$, $\mathrm{Aut}(D)$ is transitive on the set of s-arcs in D. Following [7], we say that a digraph D is *(directed)-distance transitive* if for every $s \geq 0$, $\mathrm{Aut}(D)$ is transitive on pairs (u, v) for which there is an s-arc from u to v, but no t-arc for $t < s$. Note that this implies vertex and edge transitivity, but is weaker than being highly arc transitive.

Henceforth, we shall be interested in the structure of a descendant set $\Gamma = \Gamma(\alpha)$ of a vertex α in some transitive digraph D with finite out-valency m. We will be considering this as a digraph with its full induced structure from D. We refer to α as the *root* of Γ and write $\Gamma = \Gamma(\alpha)$ to indicate that any vertex of Γ is a descendant of α. Similarly, we write Γ^i instead of $\Gamma^i(\alpha)$ for the set of vertices reachable by an i-arc starting at α and if $\beta \in \Gamma(\alpha)$, then we write $\Gamma(\beta) = \mathrm{desc}(\beta) \subseteq \Gamma(\alpha)$. Also, for $X \subseteq \Gamma$ and $s \in \mathbb{Z}$, $\Gamma^s(X) = \cup_{x\in X} \Gamma^s(x)$. It is clear that if D is highly arc transitive, then $\mathrm{Aut}(\Gamma(\alpha))$ is transitive on s-arcs in $\Gamma(\alpha)$ which start at α. Similarly, if D is distance transitive, then $\mathrm{Aut}(\Gamma(\alpha))$ is transitive on $\Gamma^n(\alpha)$ for each $n \in \mathbb{N}$.

2. Descendant sets

Throughout this section we work with digraphs Γ having the following properties:

P0 $\Gamma = \Gamma(\alpha)$ is a rooted digraph with finite out-valency $m > 0$ and $\Gamma^s(\alpha) \cap \Gamma^t(\alpha) = \varnothing$ whenever $s \neq t$.
P1 $\Gamma(u) \cong \Gamma$ for all $u \in \Gamma$.
P2 For $i \in \mathbb{N}$ the automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on Γ^i.
P3 For $i \geq 1$, $|\Gamma^i| < |\Gamma^{i+1}|$.

Note that P0 tells us that Γ is a rooted layered digraph. In [2] the author shows that Γ admits a non-trivial finite-to-one homomorphism onto a tree. In particular, if Γ has prime out-valency p, then Γ is a tree. In this section we review some of the basic results from [2] on the structure of digraphs Γ satisfying P0 to P3.

2.1. Preliminaries

By $\Gamma = \Gamma(\alpha)$ a 'rooted' digraph, we mean that for $x \in \Gamma$ there is a directed path from α to x, so x lies in Γ^i for some $i \geq 0$. We know that the in-valency of a vertex y in Γ is the size of the set $\Gamma^{-1}(y)$ of in-neighbours of y.

Lemma 2.1. *Let Γ be a digraph satisfying* P0, *$i \geq 1$ and $y \in \Gamma^i$. Then the set $\Gamma^{-1}(y)$ is contained in Γ^{i-1}.*

Proof. Let $i \geq 1$ and let $y \in \Gamma^i$. Let x be a vertex in $\Gamma^{-1}(y)$ and let γ be a directed path from α to x (γ exists since Γ is rooted) of length $j \geq 1$. Suppose $j \neq i - 1$. Then the concatenation γ' of γ with the edge (x, y) is a path of length $j + 1$ from α to y. This means that y lies in Γ^{j+1} and it follows $\Gamma^i \cap \Gamma^{j+1} \neq \varnothing$ with $i \neq j + 1$. This contradicts P0. □

Let $i \geq 1$. By property P2, the in-valency of any two vertices in Γ^i is the same. We denote this in-valency by r_i. By condition P1, for $u \in \Gamma$, the in-valency of vertices in $\Gamma^i(u)$ within $\mathrm{desc}(u)$ is equal to r_i. In particular, if $u \in \Gamma^1$, $\Gamma^i(u) \subseteq \Gamma^{i+1}$ and therefore $r_i \leq r_{i+1}$. It follows that $1 = r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots$ is an infinite non-decreasing sequence of natural numbers, which we refer to as the *in-valency sequence* of Γ. Also, by P1, we have $|\Gamma^i| = |\Gamma^i(u)|$ for any $u \in \Gamma$. This proves the following

Lemma 2.2 (*[2, Lemma 3.4]*)**.** *For $i \in \mathbb{N}$, $|\Gamma^i(u)| \leq |\Gamma^{i+1}(u)|$, for all $u \in \Gamma$.*

Note that if the out-valency m is equal to 1, then Γ is a tree. In the remainder we assume $m > 1$.

Lemma 2.3. *For $i \geq 2$, $r_i \leq m$. Equality holds if, and only if, $|\Gamma^{i-1}| = |\Gamma^i|$.*

Proof. By counting edges from Γ^{i-1} to Γ^i in two different ways, we obtain $|\Gamma^{i-1}| \times m = |\Gamma^i| \times r_i$. So

$$r_i = \frac{|\Gamma^{i-1}|}{|\Gamma^i|} \times m. \tag{1}$$

The result then follows from Lemma 2.2. □