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The confirmation of a conjecture on disjoint cycles in a graph[☆]

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ABSTRACT

Let t and k be two integers with $t \geq 5$ and $k \geq 2$. For a graph G and a vertex x of G, we use $d_G(x)$ to denote the degree of x in G. Define $\sigma_t(G)$ to be the minimum value of $\sum_{x \in X} d_G(x)$, where X is an independent set of G with $|X| = t$. This paper proves the following conjecture proposed by Gould et al. (2018). If G is a graph of sufficiently large order with $\sigma_t(G) \geq 2kt - t + 1$, then G contains k vertex-disjoint cycles.

1. Introduction

This paper considers only finite undirected simple graphs. For a simple graph G, we denote by $V(G)$, $E(G)$, $|V(G)|$ and $e(G)$ the vertex set, the edge set, the order and the number of edges of G, respectively. Simply write $|V(G)|$ as $|G|$. A set of subgraphs of G is *vertex-disjoint* (*or simply disjoint*) if no two of them have any vertex in common. The *independence number* of G is $\alpha(G)$. For a vertex $x \in V(G)$, the *neighborhood* of x in G is denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ is the degree of x in G. A complete graph of order n is denoted by K_n. *The minimum degree* of G is $\delta(G)$, and

$$\sigma_t(G) = \min\{\sum_{x \in X} d_G(x) : X \text{ is an independent set of } G \text{ with } |X| = t\}.$$

We define $\sigma_t(G) = \infty$ if $\alpha(G) \leq t - 1$. For two disjoint graphs G and H, we use mG and $G \cup H$ to denote m copies of G and the union of G and H, respectively. The graph $G + H$ is got by joining each vertex of G to each vertex of H.

In this paper, we consider degree sum conditions and the existence of disjoint cycles. Finding proper conditions for disjoint cycles is an interesting problem. In 1962 [6], Erdős and Pósa found a condition concerning the number of edges to ensure the existence of two disjoint cycles. They proved that if G is a graph of order $n \geq 6$ with $e(G) \geq 3n - 6$, then G has two disjoint cycles or $G \cong K_3 + (n-3)K_1$. In 1963 [4], Dirac gave a minimum degree condition for k disjoint triangles. He proved that for $k \geq 1$, any graph G with order $n \geq 3k$ and $\delta(G) \geq (n+k)/2$ contains k disjoint triangles. For general case, Corrádi and Hajnal proved a classical result.

Theorem 1 (*Corrádi and Hajnal [3]*)**.** *Suppose that $|G| \geq 3k$ and $\delta(G) \geq 2k$. Then G contains k disjoint cycles.*

Justesen improved Theorem 1 as follows.

Theorem 2 (*Justesen [10]*)**.** *Suppose that $|G| \geq 3k$ and $\sigma_2(G) \geq 4k$. Then G contains k disjoint cycles.*

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The degree condition in Theorem 2 is not sharp. Later, Enomoto and Wang independently improved Theorem 2 and got a sharp degree bound.

Theorem 3 (*Enomoto [5], Wang [12]*). *Suppose that $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$. Then G contains k disjoint cycles.*

Fujita, Matsumura, Tsugaki and Yamashita [7] gave a sharp degree sum condition on three independent vertices by proving the following theorem.

Theorem 4 (*Fujita et al. [7]*). *Suppose that $k \geq 2$ and $|G| \geq 3k+2$. If $\sigma_3(G) \geq 6k-2$, then G contains k disjoint cycles.*

Recently, Gould, Hirohata and Keller proposed a more general conjecture.

Conjecture 1 (*Gould et al. [8]*). *Let G be a graph of sufficiently large order. If $\sigma_t(G) \geq 2kt - t + 1$ for any two integers k and t with $k \geq 2$ and $t \geq 4$, then G contains k disjoint cycles.*

They showed that the degree sum condition of the conjecture is sharp by giving a counterexample $G = K_{2k-1} + mK_1$, where m is an integer with $m \geq t$. The only independent vertices in G are those in mK_1. Each of these vertices has degree $2k-1$. Thus $\sigma_t(G) = t(2k-1) = 2kt - t$ for any $4 \leq t \leq m$. Apparently, G does not contain k disjoint cycles as any cycle must contain two vertices of K_{2k-1}. In the same paper, they also verified that the case $t = 4$ is correct, which adds evidence for this conjecture.

In this paper, we solve Conjecture 1 for $t \geq 5$ by proving the following theorem.

Theorem 5. *Suppose that $k \geq 2$, $t \geq 5$ are two integers and $|G| \geq (2t-1)k$. If $\sigma_t(G) \geq 2kt - t + 1$, then G contains k disjoint cycles.*

Other related results about disjoint cycles in graphs and bipartite graphs have been studied, we refer the reader to see [2,9,11–15].

Remark. In the following, we introduce some useful notations. The number of components of G is $\omega(G)$. For a subgraph H of G and a vertex $x \in V(H)$, we denote $N_H(x) = N_G(x) \cap V(H)$ and $d_H(x) = |N_H(x)|$. If $S \subseteq H$, then define $d_H(S) = \sum_{x \in S} d_H(x)$ and $G[S]$ is the subgraph induced by S, and $G - S = G[V(G) - S]$. Let X, Y be two vertex-disjoint subsets or subgraphs of G, $E(X, Y)$ denotes the set of edges of G joining a vertex in X and a vertex in Y. If $X = \{x\}$, we denote $E(x, Y)$ instead of $E(\{x\}, Y)$. Let $e(X, Y) = |E(X, Y)|$ and $e(x, Y) = |E(x, Y)|$. Moreover, we write $X - x$, $X \cup x$ and $X \cap Y = x$ instead of $X - \{x\}$, $X \cup \{x\}$ and $X \cap Y = \{x\}$, respectively. We further define $(d_1, \ldots, d_n)$ with $d_1 \geq \cdots \geq d_n$ to be the *degree sequence* from X to Y if there exist n vertices $v_1, \ldots, v_n$ in X such that $e(v_i, Y) \geq d_i$ for each $1 \leq i \leq n$. A forest is a graph each of whose components is a tree. A leaf is a vertex of a forest whose degree is at most 1.

2. Lemmas

To prove Theorem 5, we make use of the following lemmas.

Let $C_1, \ldots, C_k$ be k disjoint cycles of a graph G. If $C'_1, \ldots, C'_k$ are k disjoint cycles of G and $|\cup_{i=1}^k V(C'_i)| < |\cup_{i=1}^k V(C_i)|$, then we say $\{C'_1, \ldots, C'_k\}$ *shorter cycles* than $\{C_1, \ldots, C_k\}$. We also say $\{C_1, \ldots, C_k\}$*minimal* if G does not contain k disjoint cycles $C'_1, \ldots, C'_k$ such that $|\cup_{i=1}^k V(C'_i)| < |\cup_{i=1}^k V(C_i)|$.

Lemma 1 (*[7]*). *Let k be a positive integer and let $C_1, \ldots, C_k$ be k disjoint cycles of a graph G. If $\{C_1, \ldots, C_k\}$ is minimal, then $e(x, C_i) \leq 3$ for any $x \in V(G) - \cup_{i=1}^k V(C_i)$ and for any $1 \leq i \leq k$. Furthermore, $e(x, C_i) = 3$ implies $|C_i| = 3$ and $e(x, C_i) = 2$ implies $|C_i| \leq 4$.*

In what follows, we will use Lemmas 2 and 3 to prove Lemmas 4 and 5, respectively.

Lemma 2 (*[7]*). *Suppose that F is a forest with at least two components and C is a triangle which is disjoint from F. Let x_1, x_2, x_3 be leaves of F from at least two components. If $e(\{x_1, x_2, x_3\}, C) \geq 7$, then there are two disjoint cycles in $G[F \cup C]$ or there exists a triangle C' in $G[F \cup C]$ such that $\omega(G[F \cup C] - C') < \omega(F)$.*

Lemma 3 (*[7]*). *Let C be a cycle and let T be a tree with three leaves x_1, x_2, x_3, where C and T are disjoint. If $e(\{x_1, x_2, x_3\}, C) \geq 7$, then there exist two disjoint cycles in $G[C \cup T]$ or there exists a cycle C' in $G[C \cup T]$ such that $|C'| < |C|$.*

Lemma 4. *Suppose that F is a forest with at least two components, C is a triangle disjoint from F and $t \geq 3$ is an integer. Let $x_1, x_2, \ldots, x_t$ be leaves of F from at least two components. If $e(\{x_1, x_2, \ldots, x_t\}, C) \geq 2t + 1$, then there are two disjoint cycles in $G[F \cup C]$ or there exists a triangle C' in $G[F \cup C]$ such that $\omega(G[F \cup C] - C') < \omega(F)$.*