Discrete Mathematics 341 (2018) 2912–2918

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On degree sum conditions for 2-factors with a prescribed number of cycles☆

Shuya Chiba

Applied Mathematics, Faculty of Advanced Science and Technology, Kumamoto University, 2-39-1 Kurokami, Kumamoto 860-8555, Japan

ARTICLE INFO

Article history:
Received 26 October 2017
Received in revised form 24 April 2018
Accepted 28 June 2018

Keywords:
Hamilton cycles
2-factors
Vertex-disjoint cycles
Degree sum conditions

ABSTRACT

For a vertex subset X of a graph G, let $\Delta_t(X)$ be the maximum value of the degree sums of the subsets of X of size t. In this paper, we prove the following result: Let k, m be positive integers, and let G be an m-connected graph of order $n \geq 5k - 2$. If $\Delta_2(X) \geq n$ for every independent set X of size $\lceil m/k \rceil + 1$ in G, then G has a 2-factor with exactly k cycles. This is a common extension of the results obtained by Brandt et al. (1997) and Yamashita (2008), respectively.

1. Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [4]. The independence number and the connectivity of a graph G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. For a vertex x of a graph G, we denote by $d_G(x)$ and $N_G(x)$ the degree and the neighborhood of x in G. Let $\sigma_m(G)$ be the minimum degree sum of an independent set of m vertices in a graph G, i.e., if $\alpha(G) \geq m$, then

$$\sigma_m(G) = \min\left\{\sum_{x\in X} d_G(x) : X \text{ is an independent set of } G \text{ with } |X| = m\right\};$$

otherwise, $\sigma_m(G) = +\infty$. If the graph G is clear from the context, we often omit the graph parameter G in the graph invariant. In this paper, "disjoint" always means "vertex-disjoint".

A graph having a *Hamilton cycle*, i.e., a cycle containing all the vertices of the graph, is said to be *hamiltonian*. The following degree sum condition for hamiltonicity of graphs, due to Ore (1960), is classical and well known in graph theory.

Theorem A (*Ore [15]*)**.** *Let G be a graph of order $n \geq 3$. If $\sigma_2 \geq n$, then G is hamiltonian.*

Chvátal and Erdős (1972) discovered the relationship between the connectivity, the independence number and the hamiltonicity.

Theorem B (*Chvátal, Erdős [10]*)**.** *Let G be a graph of order at least 3. If $\alpha \leq \kappa$, then G is hamiltonian.*

Bondy [2] pointed out that if a graph satisfies the Ore condition, then the graph also satisfies the Chvátal–Erdős condition, that is, Theorem B implies Theorem A.

☆ An extended abstract has been accepted in EuroComb2017, Electr. Notes Discrete Math., vol. 61, 2017, pp. 239–245.
E-mail address: schiba@kumamoto-u.ac.jp.

https://doi.org/10.1016/j.disc.2018.06.045
0012-365X/

By Theorem B, we should consider degree sum conditions for the existence of a Hamilton cycle in graphs G with $\alpha(G) \geq \kappa(G) + 1$. In fact, Bondy (1980) gave the following degree sum condition by extending Theorem B.

Theorem C (*Bondy* [3]). *Let m be a positive integer, and let G be an m-connected graph of order $n \geq 3$. If $\sigma_{m+1} > \frac{1}{2}(m+1)(n-1)$, then G is hamiltonian.*

In 2008, Yamashita [16] introduced a new graph invariant and further generalized Theorem C as follows. For a vertex subset X of a graph G with $|X| \geq t$, we define

$$\Delta_t(X) = \max\Big\{\sum_{x \in Y} d_G(x) : Y \subseteq X,\ |Y| = t\Big\}.$$

Let $m \geq t$, and if $\alpha(G) \geq m$, then let

$$\sigma_t^m(G) = \min\Big\{\Delta_t(X) : X \text{ is an independent set of } G \text{ with } |X| = m\Big\};$$

otherwise, $\sigma_t^m(G) = +\infty$. Note that $\sigma_t^m(G) \geq \frac{t}{m} \cdot \sigma_m(G)$.

Theorem D (*Yamashita* [16]). *Let m be a positive integer, and let G be an m-connected graph of order $n \geq 3$. If $\sigma_2^{m+1} \geq n$, then G is hamiltonian.*

This result suggests that the degree sum of non-adjacent "two" vertices is important for Hamilton cycles.

A *2-factor* of a graph is a spanning subgraph in which every component is a cycle, and thus a Hamilton cycle is a 2-factor with "exactly 1 cycle". In this study, we focus on 2-factors with "exactly k cycles" and consider generalizations of the above results in terms of the 2-factors. The following theorem, due to Brandt, Chen, Faudree, Gould and Lesniak (1997), is a generalization of Theorem A.

Theorem E (*Brandt et al.* [5]). *Let k be a positive integer, and let G be a graph of order $n \geq 4k - 1$. If $\sigma_2 \geq n$, then G has a 2-factor with exactly k cycles.*

In [5], the order condition is not "$n \geq 4k - 1$" but "$n \geq 4k$". However, by using a theorem of Enomoto [11] for *packing k cycles*, i.e., finding k disjoint cycles in graphs, we can improve the order condition. See the proof in [5, Lemma 1].

By considering the relation between Theorems A and E, Chen, Gould, Kawarabayashi, Ota, Saito and Schiermeyer [6] conjectured that if the order of a 2-connected graph G is large compared with only k, then the Chvátal–Erdős condition in Theorem B guarantees the existence of a 2-factor with exactly k cycles in G (see [6, Conjecture 1]). Chen et al. proved that if the order of a 2-connected graph G with $\alpha(G) = \alpha$ is sufficiently large compared with k and with the Ramsey number $r(\alpha + 4, \alpha + 1)$, then it is true. In [13], Kaneko and Yoshimoto "almost" solved the above conjecture for $k = 2$ (see also the comment after Theorem E in [6]). Another related result can be found in [7]. But, the above conjecture is still open in general. In this sense, there is a big gap between Hamilton cycles and 2-factors with exactly k (≥ 2) cycles. (For other related results about 2-factors with k cycles, we refer the reader to a survey [9].)

In this paper, by combining the techniques of the proof for hamiltonicity and the proof for 2-factors with a prescribed number of cycles, we give the following Yamashita-type condition for 2-factors with k cycles.

Theorem 1. *Let k, m be positive integers, and let G be an m-connected graph of order $n \geq 5k - 2$. If $\sigma_2^{\lceil m/k \rceil + 1} \geq n$, then G has a 2-factor with exactly k cycles.*

This theorem implies the following:

Remark 2.

- Theorem 1 is a generalization of Theorem D.
- Theorem 1 leads to the Bondy-type condition: If G is an m-connected graph of order $n \geq 5k - 2$ with $\sigma_{\lceil m/k \rceil + 1}(G) > \frac{1}{2}(\lceil m/k \rceil + 1)(n - 1)$, then G has a 2-factor with exactly k cycles. Therefore, Theorem 1 is an extension of Theorem E for sufficiently large graphs. (Recall that $\sigma_t^m(G) \geq \frac{t}{m} \cdot \sigma_m(G)$ and $\sigma_m(G) \geq \frac{m}{2} \cdot \sigma_2(G)$ for $m \geq t \geq 2$.)
- Theorem 1 leads to the Chvátal–Erdős-type condition: If G is a graph of order at least $5k - 2$ with $\alpha(G) \leq \lceil \kappa(G)/k \rceil$, then G has a 2-factor with exactly k cycles.

This implies that we can generalize results on Hamilton cycles to results on 2-factors with exactly k cycles by allowing the size of independent sets imposing degree conditions to depend on the prescribed number k.

The complete bipartite graph $K_{(n-1)/2,(n+1)/2}$ (n is odd) does not contain a 2-factor, and hence the degree condition in Theorem 1 is best possible in this sense. The order condition in Theorem 1 comes from our proof techniques. Similar to the situation for the proof of Theorem E, we will use the order condition only for packing k cycles (see Lemma 5 and the proof of Theorem 1 in Section 3). The complete bipartite graph $K_{2k-1,2k-1}$ shows that $n \geq 4k - 1$, in a sense, is necessary. In the last section (Section 4), we note that "$n \geq 5k - 2$" can be replaced with "$n \geq 4k - 1$" for the Bondy-type condition (and the Chvátal–Erdős-type condition) in Remark 2.