Discrete Mathematics 341 (2018) 2544–2552

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On packing of rectangles in a rectangle

Antal Joós

University of Dunaújváros, Táncsics Mihály utca 1/A, Dunaújváros, 2400, Hungary

ARTICLE INFO

Article history:
Received 4 July 2017
Received in revised form 15 May 2018
Accepted 3 June 2018

Keywords:
Packing
Rectangle
Smallest area

ABSTRACT

It is known that $\sum_{i=1}^{\infty} 1/(i(i+1)) = 1$. In 1968, Meir and Moser (1968) asked for finding the smallest ϵ such that all the rectangles of sizes $1/i \times 1/(i+1)$, $i \in \{1, 2, \ldots\}$, can be packed into a square or a rectangle of area $1 + \epsilon$. First we show that in Paulhus (1997), the key lemma, as a statement, in the proof of the smallest published upper bound of the minimum area is false, then we prove a different new upper bound. We show that $\epsilon \le 1.26 \cdot 10^{-9}$ if the rectangles are packed into a square and $\epsilon \le 6.878 \cdot 10^{-10}$ if the rectangles are packed into a rectangle.

1. Introduction

We can read in [10] about the following problem. Since $\sum_{i=1}^{\infty} 1/(i(i+1)) = 1$, it is reasonable to ask whether the set of rectangles of sizes $1/i \times 1/(i+1)$, $i \in \{1, 2, \ldots\}$, can be packed into a square or rectangle of area 1. Failing that, find the smallest ϵ_S and ϵ_R such that the above rectangles can be packed in a square of area $1+\epsilon_S$ or in a rectangle of area $1+\epsilon_R$. The problem also appears in [3,5] and [4].

Meir and Moser [10] showed that the square of side length $1 + 1/30$ contains all the rectangles of sizes $1/i \times 1/(i+1)$, $i \in \{1, 2, \ldots\}$, which shows that $\epsilon_R \le \epsilon_S < 0.0678$. Jennings [8] proved that the square of side length 133/132 contains these rectangles, after Jennings [9] proved that the square of side length 204/203 contains these rectangles, which shows that $\epsilon_S < 0.009877$. Bálint [1] and [2] proved that the square of side length 501/500 contains these rectangles, which shows that $\epsilon_S \le 0.004\,004$. Bálint [1] and [2] proved that a rectangle of area 1.002 4 contains these rectangles which shows that $\epsilon_R \le 0.002\,4$. Paulhus [11] tried to give an estimate for ϵ_R and ϵ_S but the proof of the lemma in [11] is incorrect and the statement of the lemma is not true. We show, that the lemma of Paulhus is not true thus the best known results are $\epsilon_S \le 0.004\,004$ and $\epsilon_R \le 0.002\,4$.

2. Notation

In this paper the width of a rectangle will always refer to the shorter side and the height will always refer to the longer side of the rectangle. The width (height, resp.) of the rectangle R is denoted by $w(R)$ ($h(R)$, resp.). The rectangle of dimensions $1/i \times 1/(i+1)$ is referred by (the rectangle) P_i (or simply i). To avoid confusion we will call the unused rectangles inside the unit square boxes.

3. The lemma and the algorithm of paulhus

Let $l_P = 0.000\,0188\,31$ and $n_0 = 2\,820\,079\,889$. Paulhus used computer to pack the rectangles of sizes $1/i \times 1/(i+1)$, $i \in \{1, 2, \ldots, 10^9\}$, and he realized that there is an unfilled box of dimensions $l_P \times l_P$. The lemma of Paulhus said that the

E-mail address: joosa@uniduna.hu.

https://doi.org/10.1016/j.disc.2018.06.007
0012-365X/

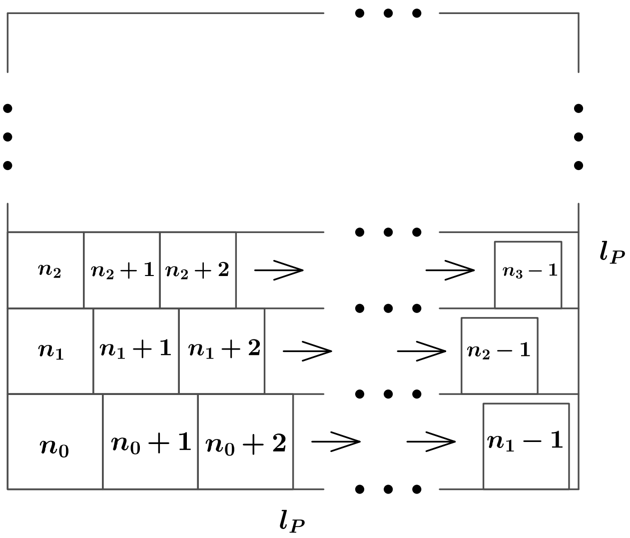

Fig. 1. The squares in S_P.

rectangles $n_0, n_0+1, \ldots$ can be packed into the empty box of sizes $l_P \times l_P$. He packed the rest of the rectangles – from 10^9+1 to $n_0 - 1$ – in a rectangle of dimensions $1/(10^9+1) \times 1$ and placed it on the top of the unit square.

First, we show, that the lemma of Paulhus is not true. Let S_P be the square of side length l_P. Paulhus overestimated the area of the rectangle P_i by assuming it is the square of side length $1/i$ and stated that the rectangles (squares) $n_0, n_0+1, \ldots$ fit in the square S_P if the rectangles (squares) go from n_{i-1} to $n_i - 1$ in the ith row in S_P where $n_i = \lfloor n_{i-1}(1+l_P) \rfloor$ for $i \geq 1$ (Fig. 1).

We have $n_i \leq n_1(1+l_P)^{i-1}$ for $i \geq 1$ and

$$\sum_{i=0}^{\infty} \frac{1}{n_i} = \frac{1}{n_0} + \sum_{i=1}^{\infty} \frac{1}{n_i} \geq \frac{1}{n_0} + \sum_{i=1}^{\infty} \frac{1}{n_1(1+l_P)^{i-1}}$$
$$= \frac{1}{n_0} + \frac{1}{n_1} \sum_{i=0}^{\infty} \frac{1}{(1+l_P)^i} = \frac{1}{n_0} + \frac{1}{n_1} \frac{1}{1 - \frac{1}{1+l_P}} \tag{1}$$
$$= 0.000\,018\,831\,000\,002\,7\ldots > l_P,$$

that is the squares of sides $1/n_0, 1/n_1, \ldots$ do not fit in the square S_P thus the best known estimates are $\epsilon_S \leq 0.004\,004$ and $\epsilon_R \leq 0.002\,4$.

Remark 1.

A closer look of page 156 in [11] reveals that Paulhus uses the following rules:

"*Rule* 1. Place the rectangle P_n in a corner of the smallest width box into which it will fit under either orientation. If P_n fits equally well in two or more boxes choose to place it in the box with the shortest height.

Rule 2. After placing a rectangle in the corner of a box, always cut the remaining area into two rectangular pieces by cutting from the corner of the rectangle to the longer side of the original box".

After placing rectangle P_1 the possible arrangements of P_2 are shown in Figs. 2 and 3. Assuming the arrangement in Fig. 2 is the right arrangement, as in [11], the possible arrangements of P_3 are shown in Figs. 4 and 5.

The orientation is important. Let us change Rule 1 in the following way.

Rule 1'. Place the rectangle P_i in a corner of a smallest width box B_i into which it will fit so that $w(B_i) \geq h(P_i)$. Place the rectangle P_i into B so that the sides of lengths $w(B_i)$ and $h(P_i)$ are parallel to each other.

Observe Figs. 2 and 4 are the correct arrangements if Rule 1' is used.

If Rule 1' and Rule 2 were used, then after placing the rectangles $P_1, \ldots, P_{10^9}$ into the unit square the largest unfilled box had a height and width each greater than $0.000\,018\,568$, which is less than l_P. Thus the algorithm of the third problem in [11] was not determined uniquely and we cannot check the result of Paulhus.

Remark 2. In his proof Paulhus uses the expression $n_{i+1} = n_i \lfloor (1+l_P) \rfloor$ instead of the correct expression $n_{i+1} = \lfloor n_i(1+l_P) \rfloor$ but – obviously – this is only a misspelling.. A more deeper look into the proof shows that the real mistake is the opposite