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A condition for assured 3-face-colorability of infinite plane graphs with a given spanning tree

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ABSTRACT

Given an infinite leafless tree drawn on the plane, we ask whether or not one can add edges between the vertices of the tree obtaining a non-3-face-colorable graph. We formulate a condition conjectured to be necessary and sufficient for this to be possible. We prove that this condition is indeed necessary and sufficient for trees with maximal degree 3, and that it is sufficient for general trees. In particular, we prove that every infinite plane graph with a spanning binary tree is 3-face-colorable.

1. Introduction

The Four Color Theorem [1] states that every finite planar graph is both vertex colorable and face colorable with 4 colors. (In this paper we are mainly concerned with face coloring.) Using the Erdős–de Bruijn Theorem [2], we can prove the same for infinite planar graphs. However, coloring with 3 colors is not always possible and determining whether a given graph is 3-colorable is a hard task [3]. Our aim here is to study 3-face-coloring of planar graphs by means of a given spanning tree.

Two edges in a plane graph G are called *G-adjacent* if they have a common vertex and they are adjacent in the cyclic order of the edges around this vertex. Two faces are called *Touching* if they have a common edge. A tree is called *leafless* if its minimal degree is at least 2. A drawing of a graph on the plane is called *diverging* if there is no infinite sequence of vertices drawn on points converging to a point on the plane. In this paper we restrict our discussion to leafless diverging trees. We say that a plane graph G is called a *fouring* of a plane tree T if T is a spanning tree of G, i.e., G is obtained from T by adding edges between existing vertices without changing the locations of the vertices or the drawing of the existing edges, and in addition, G is not 3-face-colorable. If T has a fouring then we say that T is *fourable*. Otherwise, we say that T is *unfourable*. The main aim of this paper is to characterize the leafless diverging plane trees that are fourable. For trees with maximal degree 3, we prove that a leafless diverging plane tree is fourable if and only if it has two vertices with degree 2 with an even distance between them. The following definition attempts to generalize this property to trees with vertices of degree more than 3.

Let T be a leafless diverging plane tree and let S be a finite subtree of T. We say that S is a *simply splittable* subtree of T if

(1) Every vertex of S is in at most one edge of T which is not in S.
(2) For every vertex v of S, it is possible to write $E(S)$ as the disjoint union of the edge sets of subtrees of S, in each of which either v appears in exactly one edge, or v appears in exactly two edges, which are T-adjacent, and each of these subtrees has an even number of edges. We refer to each of these subtrees as *v-even*.

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Note that a simply splittable subtree must have an even number of edges. We say that S is *splittable* if there exists a set of edges $X \subset E(S)$ such that S/X is a simply splittable subtree of T/X. (Where here / denotes edge contraction. In particular, a simply splittable subtree is splittable, since X is allowed to be empty.)

Our main conjecture is

Conjecture 1. *A leafless diverging plane tree is fourable if and only if it has a splittable subtree.*

In this paper we prove the conjecture for trees with maximal degree 3

Theorem 1. *Let T be a diverging plane tree, where all vertices have degree 2 or 3, then T is fourable if and only it has a splittable subtree.*

In addition, we prove one direction for general trees

Theorem 2. *A diverging tree with a splittable subtree is fourable.*

For the other direction, we prove a weaker result:

Theorem 3. *Every fourable diverging plane tree T has a subtree S in which every vertex is in at most one edge not belonging to S.*

2. Trees with maximal degree 3

In this section we prove Theorem 1. Throughout this section we fix a diverging plane tree T, where all vertices have degree 2 or 3.

Lemma 1. *The tree T has a splittable subtree if and only if T has two vertices with degree 2 of even distance between them.*

Proof. If T has two vertices with degree 2 of even distance between them then clearly the path between them is simply splittable.

Now let us prove the other direction. Let S be a splittable subtree of T. Then the leaves of S have degree 2 in T. If S has three or more leaves then some two of them have even distance and we are done. Thus we may assume S has only two leaves u and v, in other words S is a path between u and v. We may also assume that all vertices of T other than u and v have degree 3.

We now need to show that S has an even number of edges. Since we assume that all vertices of T other than u and v have degree 3, each vertex of S is in exactly one edge of T not in S. Therefore, any contraction of one or more edges of S will result in a vertex which is in more than one edge of T not in S. This means that in fact S is simply splittable in T. Hence the number of edges in S is even, and this is the distance between the two vertices u, v of degree 2. □

Theorem 4. *If T has two vertices with degree 2 of even distance between them, then T is fourable.*

Proof. Let x, y be two vertices degree 2 of even distance between them. We choose such vertices with minimal distance between them. Let $x = v_0, v_1, \dots, v_{2k} = y$ be the path between x and y. By the minimality of k, all of $v_1, v_2, \dots, v_{2k-1}$ have degree 3, except possibly to one vertex v_j where j is odd. If such a vertex exists, we start the construction of our fouring by adding an edge between v_j to some vertex outside the path. Now all of $v_1, \dots, v_{2k-1}$ have degree 3. Next we add the edges of the path $v_0, v_2, v_4, \dots, v_{2k}$. Note that we can do this while keeping the planarity. We call the obtained graph G. For each $i = 0, \dots, k-1$, denote by D_i the triangle between v_{2i}, v_{2i+1} and v_{2i+2}. For each $i = 1, \dots, k-1$, the vertex v_{2i} has degree 5 in G and therefore appears in 5 faces. Two of these faces are D_i and D_{i-1}. The other three faces are as follows:

- One face touches D_{i-1} but not D_i. We denote it by A_i.
- One face touches both D_{i-1} and D_i. We denote it by B_i.
- One face touches D_i but not D_{i-1}. We denote it by C_i.

Note that equalities, such as $C_{i-1} = A_i$ or $B_{i-1} = B_i$ etc. might hold. We also denote by C_0 the face touching D_0 other than A_1 and B_1, and we denote by A_k, B_k the two faces in which v_{2k} appears other than D_{k-1}.

Assume for contradiction that f is some 3-face-coloring of G. We shall prove by induction that $f(A_i) = f(B_i)$ for every $i = 1, \dots, k$. This will yield a contradiction since A_k and B_k are touching. The base of the induction follows from the fact that each of A_1 and B_1 touches both C_0 and D_0. Now assume $f(A_{i-1}) = f(B_{i-1}) = c$ for some $i = 2, \dots, k$ and our aim is to prove $f(A_i) = f(B_i)$. If $B_i \neq B_{i-1}$ and $A_i \neq B_{i-1}$ then the vertex v_{2i} cannot be in the boundary of B_{i-1}, and hence the only edge at which B_{i-1} touches D_{i-1} is $\{v_{2i-2}, v_{2i-1}\}$. This means that the two faces touching D_{i-1} at edges $\{v_{2i-1}, v_{2i}\}$ and $\{v_{2i-2}, v_{2i}\}$ (not necessarily respectively) are A_i and B_i. This implies that each of A_i and B_i touches either A_{i-1} or B_{i-1} and thus both $f(A_i)$ and $f(B_i)$ should be the color different than c and $f(D_{i-1})$. If $B_i = B_{i-1}$ then A_i touches C_{i-1}. Hence $f(A_i)$ should be the color different than $f(C_{i-1})$ and $f(D_{i-1})$, which is c. A similar argument holds where $A_i = B_{i-1}$. □