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The degree-diameter problem for circulant graphs of degrees 10 and 11

Robert R. Lewis

Department of Mathematics and Statistics, The Open University, Milton Keynes, UK

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ABSTRACT

This paper considers the degree-diameter problem for undirected circulant graphs. For degrees 10 and 11, newly discovered families of circulant graphs of arbitrary diameter are presented which are largest known and are conjectured to be extremal. They are also the largest-known Abelian Cayley graphs of these degrees. For each such family, the order of every graph in the family is defined by a quintic polynomial function of the diameter which is specific to the family. The elements of the generating set for each graph are similarly defined by a set of polynomials in the diameter. The existence of the graphs in the degree 10 families has been proved for all diameters. These graphs are consistent with a conjecture on the order of extremal Abelian Cayley and circulant graphs of any degree and diameter.

1. Introduction

The *degree-diameter problem* is the problem of finding graphs with the largest possible number of vertices $n(d, k)$ for a given maximal degree d and diameter k. Such graphs are called *extremal* graphs. From the literature, it is seen that this problem has been tackled for undirected, directed and mixed graphs. In addition to the general case, various subproblems have also been explored, including vertex-transitive graphs and Cayley graphs. For a general background on the degree-diameter problem, see the comprehensive survey by Miller and Širáň [6] and the tables of largest-known graphs on the CombinatoricsWiki website [2].

Only in relatively few cases are the largest-known graphs believed to be extremal, typically restricted to degree 3 for small diameter or diameter 2 for small degree. Circulant graphs, which are Cayley graphs of cyclic groups and therefore highly structured, are a noteworthy exception. In this paper, $CC(d, k)$ denotes the order of an extremal undirected circulant graph of degree d and diameter k, and $AC(d, k)$ similarly for Abelian Cayley graphs. $L_{CC}(d, k)$ is the order of the largest-known circulant graph when the extremal order is unknown. For even degree d, if $L_{CC}(d, k)$ is even we additionally define $L_{OC}(d, k)$ to be the order of the largest-known graph of odd order.

Infinite families of extremal undirected circulant graphs have been identified and proven extremal by various authors for degrees $d = 2, 3, 4$ and 5, with order $CC(d, k)$ defined by a polynomial in the diameter k for any diameter [3]. Similar families of largest-known circulant graphs of order $L_{CC}(d, k)$ were discovered for degree 6 by Monakhova in 2003 [7] and independently by Dougherty and Faber, also for degree 7, in 2004 [3].

It happens that $CC(2, k)$, $CC(4, k)$ and $L_{CC}(6, k)$ are all odd for any diameter k. Families of largest-known degree 8 graphs of odd order $L_{OC}(8, k)$ were discovered by Monakhova in 2013 and conjectured to be extremal [8]. However, Monakhova had limited her search to odd order graphs because of a 1994 paper by Muga in which he mistakenly claimed that any extremal circulant graph of even degree and arbitrary diameter has odd order [9]. The argument was flawed, and the smallest

E-mail address: robert.lewis@open.ac.uk.

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counterexample is the extremal graph of degree 8 and diameter 3, which has order 104. Families of largest-known graphs of degrees 8 and 9 were discovered by the author in 2014 [4]. The degree 8 graphs have even order for any diameter $k \geq 3$.

These circulant graph families of degrees 6 to 9 all remain largest known to date. They have been proven by computer search to be extremal for small diameters, and they are conjectured to be extremal for all larger diameters. For details of their order, generating sets and range of proven extremality, see [4]. The main result of this paper is the construction of infinite families of largest-known circulant graphs of degrees 10 and 11, with order and generating sets defined by polynomials in the diameter, which are conjectured to be extremal.

A circulant graph is so called because its adjacency matrix is a circulant matrix. As mentioned, a circulant graph X of order n may also be viewed as a Cayley graph whose vertices are the elements of the cyclic group $\mathbb{Z}_n$. Two vertices i, j are connected by an arc (i, j) if and only if $j - i$ is an element of C, a subset of $\mathbb{Z}_n \setminus 0$, called the *connection set*. We may also denote the graph by $X(\mathbb{Z}_n, C)$. If C is closed under additive inverses then X is an undirected graph, and in this paper we will only consider undirected graphs. By definition such a graph is regular, with the degree d of each vertex equal to the cardinality of C. If n is odd, $\mathbb{Z}_n \setminus 0$ has no elements of order 2. Therefore C has even cardinality, say $d = 2f$, and comprises f complementary pairs of elements, with one of each pair strictly between 0 and $n/2$. Any set of size f containing exactly one element from each pair is sufficient to uniquely determine the connection set and is called a *generating set*. Without loss of generality in this paper we will choose the generating set which is comprised of the f elements of C between 0 and $n/2$ as the canonical generating set G for X. If n is even, $\mathbb{Z}_n \setminus 0$ has just one element of order 2, namely $n/2$. In this case, C comprises f complementary pairs of elements, as for odd n, with or without the addition of the involutory element $n/2$. If C has odd cardinality, so that $d = 2f + 1$, then the value of its involutory element is defined by the value of n. Therefore for a circulant graph of given order and degree, its connection set C is completely defined by specifying its generating set G. The cardinality of the connection set is equal to the degree d of the graph, and the cardinality of the generating set, f, is defined to be the dimension of the graph.

Clearly, if every element of a generating set is multiplied by a constant factor that is co-prime with the order of the graph, then the resultant set will also be a generating set of a graph which is isomorphic to the first. Therefore the isomorphism class of a circulant graph could have a number of different generating sets. Not all isomorphism classes of circulant graphs have a primitive generating set (where one of the generators is 1). An example of an extremal circulant graph with no primitive generating set is the graph with degree 9, diameter 2, order 42 and generating set $\{2, 7, 8, 10\}$. For simplicity, where a graph has at least one primitive generating set, we will only consider the primitive sets. In fact, it emerges that every family of extremal or largest-known circulant graphs so far discovered has at least one primitive generating set.

2. Conjectured order of extremal Abelian Cayley and circulant graphs of any degree and diameter

We briefly review general upper and lower bounds for the order of extremal Abelian Cayley and circulant graphs of arbitrary degree d and diameter k. For Abelian Cayley graphs, and thus in particular for circulant graphs, an upper bound that is much sharper than the general Moore bound was established for even degree by Wong and Coppersmith in 1974 [11] and further sharpened by Muga in 1994 [9]. Dougherty and Faber developed an equivalent upper bound for odd degree in 2004 [3].

For positive integers f, k, we define $S_{f,k}$ to be the set of elements of $\mathbb{Z}^f$ (the f-dimensional direct product of $\mathbb{Z}$ with itself) which can be expressed as a word of length at most k in the canonical generators $\mathbf{e}_i$ of $\mathbb{Z}^f$, taken positive or negative. Equivalently, $S_{f,k}$ is the set of points in $\mathbb{Z}^f$ distant at most k from the origin under the ℓ^1 (Manhattan) metric: $S_{f,k} = \{(x_1, \ldots, x_f) \in \mathbb{Z}^f : |x_1| + \cdots + |x_f| \leq k\}$. Within the literature on coding theory and tiling problems, $S_{f,k}$ is often called the f-dimensional Lee sphere of radius k, although it appears more diamond-like than spherical, having the form of a regular dual f-cube.

For an Abelian Cayley graph of degree d and diameter k, and corresponding dimension $f = \lfloor d/2 \rfloor$, Muga's and Dougherty and Faber's upper bounds are defined by:

$$M_{AC}(d, k) = \begin{cases} |S_{f,k}| & \text{for even } d \\ |S_{f,k}| + |S_{f,k-1}| & \text{for odd } d, \end{cases}$$

where, by [10],

$$|S_{f,k}| = \sum_{i=0}^{f} 2^i \binom{f}{i} \binom{k}{i}.$$

For even or odd degree, this is a polynomial in k of order f:

$$M_{AC}(d, k) = \begin{cases} \dfrac{2^f}{f!} k^f + \dfrac{2^{f-1}}{(f-1)!} k^{f-1} + O(k^{f-2}) & \text{for even } d \\ \dfrac{2^{f+1}}{f!} k^f \qquad\qquad\qquad\qquad + O(k^{f-2}) & \text{for odd } d. \end{cases}$$