Discrete Mathematics 341 (2018) 2925–2937

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Trimming of metric spaces and the tight span

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ARTICLE INFO

Article history:
Received 28 November 2017
Accepted 6 July 2018

Keywords:
Tight span
Trimming

ABSTRACT

We use the trimming transformations to study the tight span of a metric space.

1. Introduction

The theory of tight spans due to J. Isbell [4] and A. Dress [2] embeds any metric space X in a hyperconvex metric space $T(X)$ called the tight span of X. In this paper we split $T(X)$ as a union of two metric subspaces $\tau = \tau(X)$ and $\overline{C} = \overline{C(X)}$. The space τ is the tight span of a certain quotient $\overline{X_\infty}$ of X. The space $\overline{C}$ is a disjoint union of metric trees which either do not meet τ or meet τ at their roots lying in $\overline{X_\infty} \subset \tau$. In this picture, the original metric space $X \subset T(X)$ consists of the tips of the branches of the trees.

The construction of τ and $\overline{C}$ uses the trimming transformations of metric spaces studied in [6] for finite metric spaces. In the present paper – essentially independent of [6] – we discuss trimming for all metric spaces and introduce related objects including the subspaces τ and $\overline{C}$ of $T(X)$. Our main theorem says that $\tau \cup \overline{C} = T(X)$ and $\tau \cap \overline{C}$ is the set of the roots of the trees forming $\overline{C}$.

2. Trim pseudometric spaces

2.1. Pseudometrics

We recall basics on metric and pseudometric spaces. Set $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$. A *pseudometric space* is a pair consisting of a set X and a mapping $d : X \times X \to \mathbb{R}_+$ (the *pseudometric*) such that for all $x, y, z \in X$,

$$d(x,x) = 0, \quad d(x,y) = d(y,x), \quad d(x,y) + d(y,z) \geq d(x,z).$$

A pseudometric space (X, d) is a *metric space* (and d is a *metric*) if $d(x, y) > 0$ for all distinct $x, y \in X$.

A map $f : X \to X'$ between pseudometric spaces (X, d) and (X', d') is *distance preserving* if $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$ and is *non-expansive* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$. Pseudometric spaces X, X' are *isometric* if there is a distance preserving bijection $X \to X'$. We call distance preserving maps between metric spaces *metric embeddings*; they are always injective.

Each pseudometric space (X, d) carries an equivalence relation $\sim_d$ defined by $x \sim_d y$ if $d(x, y) = 0$ for $x, y \in X$. The quotient set $\overline{X} = X/\sim_d$ carries a metric $\overline{d}$ defined by $\overline{d}(\overline{x}, \overline{y}) = d(x, y)$ where x, y are any points of X and $\overline{x}, \overline{y} \in \overline{X}$ are their equivalence classes. The metric space $(\overline{X}, \overline{d})$ is the *metric quotient* of (X, d). Any distance preserving map from X to a metric space Y expands uniquely as the composition of the projection $X \to \overline{X}$ and a metric embedding $\overline{X} \hookrightarrow Y$.

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https://doi.org/10.1016/j.disc.2018.07.005
0012-365X/

2.2. Trim spaces

Given a set X and a map $d : X \times X \to \mathbb{R}$, we use the same symbol d for the map $X \times X \times X \to \mathbb{R}$ carrying any triple $x, y, z \in X$ to

$$d(x, y, z) = \frac{d(x, y) + d(x, z) - d(y, z)}{2}.$$

The right-hand side is called the Gromov product, see, for instance, [1].

A pseudometric d in a set X determines a function $\underline{d} : X \to \mathbb{R}_+$ as follows: if $\mathrm{card}(X) = 1$, then $\underline{d} = 0$; if X has two points x, y, then $\underline{d}(x) = \underline{d}(y) = d(x, y)/2$; if $\mathrm{card}(X) \geq 3$, then for all $x \in X$,

$$\underline{d}(x) = \inf_{y,z \in X \setminus \{x\}, y \neq z} d(x, y, z).$$

If $\underline{d}(x) = 0$ for all $x \in X$, then we say that the pseudometric space (X, d) is *trim*.

Following K. Menger [5], we say that a point x of a pseudometric space (X, d) *lies between* $y \in X$ and $z \in X$ if $d(y, z) = d(x, y) + d(x, z)$. A simple sufficient condition for (X, d) to be trim says that each point $x \in X$ lies between two distinct points of $X \setminus \{x\}$.

2.3. Examples

A pseudometric space having only one point is trim. A pseudometric space having two points is trim iff the distance between these points is equal to zero. More generally, any set with zero pseudometric is trim. We give two examples of trim metric spaces:

(a) the set of words of a fixed finite length in a given finite alphabet with the Hamming distance between words defined as the number of positions at which the letters of the words differ;

(b) a subset of a Euclidean circle $C \subset \mathbb{R}^2$ meeting each half-circle in C in at least three points; here the distance between two points is the length of the shorter arc in C connecting these points.

3. Drift and trimming

3.1. The drift

Given a function $\delta : X \to \mathbb{R}$ on a metric space (X, d), we define a map $d_\delta : X \times X \to \mathbb{R}$ as follows: for $x, y \in X$,

$$d_\delta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ d(x, y) - \delta(x) - \delta(y) & \text{if } x \neq y. \end{cases}$$

We say that d_δ is obtained from d by a *drift*. The idea behind the definition of d_δ is that each point $x \in X$ is pulled towards all other points of X by $\delta(x)$.

Lemma 3.1. *If $\delta \leq \underline{d}$ (i.e., if $\delta(x) \leq \underline{d}(x)$ for all $x \in X$), then d_δ is a pseudometric in X. Moreover, if* $\mathrm{card}(X) \geq 3$ *or* $\mathrm{card}(X) = 2$, $\delta = const$, *then $\underline{d_\delta} = \underline{d} - \delta$.*

Proof. We first prove that for any distinct $x, y \in X$,

$$\underline{d}(x) + \underline{d}(y) \leq d(x, y). \tag{3.1.1}$$

If $X = \{x, y\}$, then (3.1.1) follows from the definition of $\underline{d}$. If $\mathrm{card}(X) \geq 3$, pick any $z \in X \setminus \{x, y\}$. Then $\underline{d}(x) \leq d(x, y, z)$ and $\underline{d}(y) \leq d(y, x, z)$. So,

$$\underline{d}(x) + \underline{d}(y) \leq d(x, y, z) + d(y, x, z) = d(x, y).$$

We now check that $d_\delta : X \times X \to \mathbb{R}$ is a pseudometric. Clearly, d_δ is symmetric and, by definition, $d_\delta(x, x) = 0$ for all $x \in X$. Formula (3.1.1) and the assumption $\delta \leq \underline{d}$ imply that for any distinct $x, y \in X$,

$$d_\delta(x, y) = d(x, y) - \delta(x) - \delta(y) \geq \underline{d}(x) - \delta(x) + \underline{d}(y) - \delta(y) \geq 0.$$

To prove the triangle inequality for d_δ we rewrite it as $d_\delta(x, y, z) \geq 0$ for any $x, y, z \in X$. If $x = y$ or $x = z$, then $d_\delta(x, y, z) = 0$; if $y = z$, then $d_\delta(x, y, z) = d_\delta(x, y) \geq 0$. Finally, if x, y, z are pairwise distinct, then

$$d_\delta(x, y, z) = d(x, y, z) - \delta(x) \geq \underline{d}(x) - \delta(x) \geq 0. \tag{3.1.2}$$

The equality $\underline{d_\delta}(x) = \underline{d}(x) - \delta(x)$ for all $x \in X$ follows from (3.1.2) if $\mathrm{card}(X) \geq 3$ and from the definitions if $\mathrm{card}(X) = 2$ and $\delta = const$. $\square$