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Reconfiguration graphs of shortest paths

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ABSTRACT

For a graph G and$a, b \in V(G)$, the shortest path reconfiguration graph of G with respect to a and$b$ is denoted by $S(G, a, b)$. The vertex set of $S(G, a, b)$ is the set of all shortest paths between a and$b$ in G. Two vertices in $V(S(G, a, b))$ are adjacent, if their corresponding paths in G differ by exactly one vertex. This paper examines the properties of shortest path graphs. Results include establishing classes of graphs that appear as shortest path graphs, decompositions and sums involving shortest path graphs, and the complete classification of shortest path graphs with girth 5 or greater. We include an infinite family of well structured examples, showing that the shortest path graph of a grid graph is an induced subgraph of a lattice.

1. Introduction

Many important problems have more than one feasible solution. The reconfiguration question is to determine whether it is possible to transform one feasible solution to a problem into a target feasible solution in a step-by-step manner (via a reconfiguration rule), such that each intermediate solution is also feasible. For example, there may be many feasible solutions for how to a properly vertex color a specific graph (proper colorings are the feasible solutions). A proper coloring reconfiguration problem is: Given a graph G and two proper colorings C_1 and C_2, is it possible to transform C_1 to C_2 by re-coloring one vertex at a time (the reconfiguration rule) while keeping the coloring proper at each step? Such transformations can be studied via the reconfiguration graph, in which the vertices represent the feasible solutions and there is an edge between two vertices when it is possible to get from one feasible solution to another in one application of the reconfiguration rule. Reconfiguration is a very lively area of current study. Recent papers about reconfiguration include some on vertex coloring [3,5–8], independent sets [15,16,18], matchings [16], list-colorings [17], matroid bases [16], and subsets of a (multi)set of numbers [10], among others. This paper concerns the reconfiguration of shortest paths in a graph.

Definition 1. Let G be a graph with distinct vertices a and b. The *shortest path graph* of G with respect to a and b is the graph $S(G, a, b)$ in which every vertex U corresponds to a shortest path in G between a and b, and two vertices $U, W \in V(S(G, a, b))$ are adjacent if and only if their corresponding paths in G differ in exactly one vertex.[1]

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1 The shortest path graph is denoted by SP(G, a, b) in [4].

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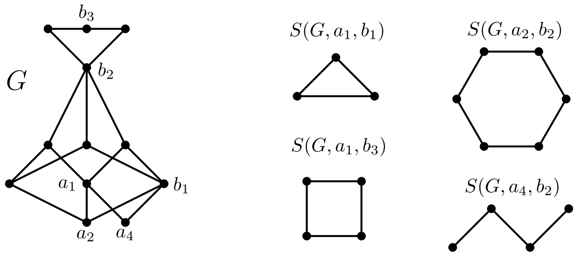

Fig. 1. Base graph G (left) with several shortest path graphs (right).

One fundamental question in reconfiguration is to determine the computational complexity of determining if two feasible solutions can be reconfigured from one to another and many papers in this area focused on that, see for example [4,5,16,18]. The graph theory community has expanded this topic to examine the structure of the reconfiguration graph in many contexts, including reconfiguration for vertex coloring [2,9,11], and domination [14]. Reconfiguration of shortest paths is unusual from a computational standpoint because, although the complexity of finding a shortest path in a graph is polynomial, the corresponding reconfiguration problem is PSPACE complete [18]. With this as background, the focus of our work is on the structure of shortest path graphs, rather than on algorithms. Our main goal is to understand which graphs occur as shortest path graphs.

Some definitions and notations are provided in Section 2. Section 3 contains some useful properties and examples. In particular, we show that paths and complete graphs are shortest path graphs. In Section 4 we show that the family of shortest path graphs is closed under disjoint union and under Cartesian products. We establish a decomposition result which suggests that, typically, 4-cycles are prevalent in shortest path graphs. Thus, we would expect the structure of shortest path graphs containing no 4-cycles to be rather simple. This is substantiated in Section 5, where we give a remarkably simple characterization of shortest path graphs with girth 5 or greater. In the process of establishing this characterization, we show that the presence of an induced claw or k-cycle implies that other, specific, induced subgraphs must be present. As a consequence, we determine precisely which cycles are shortest path graphs; that the claw, by itself, is not a shortest path graph; and that a tree cannot be a shortest path graph unless it is a path.

In contrast to the simple structure of shortest path graphs of girth 5 or greater, those of smaller girth exhibit a rich structure. We begin the exploration of such graphs in Section 6 by considering a specific family of these that showcase some of the difficulty of characterizing shortest path graphs in general. Specifically, we establish that the shortest path graph of a grid graph is an induced subgraph of the lattice. A particularly nice special case is that the shortest path graph of the hypercube Q_n with respect to two diametric vertices is a Cayley graph on the symmetric group S_n.

2. Preliminaries

We consider only simple graphs, G, with vertex set $V(G)$ and edge set $E(G)$. When two vertices x, y are adjacent we denote this as $x \sim y$. Let G be a graph with distinct vertices a and b. A shortest a, b-path in G is a path between a and b of length $d_G(a, b)$. When it causes no confusion, we write $d(a, b)$ to mean $d_G(a, b)$. We often refer to a shortest path as a geodesic and to a shortest a, b-path as an a, b-geodesic. Note that any subpath of a geodesic is a geodesic.

If the paths of G corresponding to two adjacent vertices U, W in $S(G, a, b)$ are $av_1 \cdots v_{i-1}v_iv_{i+1} \cdots v_pb$ and $av_1 \cdots v_{i-1}v'_iv_{i+1} \cdots v_pb$, we say that U and W differ in the ith index, or that i is the *difference index* of the edge UW. We call the graph G the *base graph* of $S(G, a, b)$, and we say that a graph H is a shortest path graph, if there exists a graph G with $a, b \in V(G)$ such that $S(G, a, b)$ is isomorphic to H, denoted as $S(G, a, b) \cong H$. Several examples are given in Fig. 1. With a slight abuse of notation, a label for a vertex in the shortest path graph will often also represent the corresponding path in its base graph. To avoid confusion between vertices in G and vertices in $S(G, a, b)$, throughout this paper, we will use lower case letters to denote vertices in the base graph, and upper case letters to denote vertices in $S(G, a, b)$.

It can easily be seen that several base graphs can have the same shortest path graph. For example, if $e \in E(G)$ and e is an edge not in any a, b-geodesic, then $S(G, a, b) \cong S(G \setminus e, a, b)$. To this end, we define the *reduced graph,* (G, a, b), to be the graph obtained from G by deleting any edge or vertex that does not occur in any a, b-geodesic, and contracting any edge that occurs in all a, b-geodesics. If the reduced graph (G, a, b) is again G, then G is called a *reduced graph* with respect to a, b. We may omit the reference to a, b when it is clear from context.

We conclude this section with a review of some basic definitions. If G_1 and G_2 are graphs, then $G_1 \cup G_2$ is defined to be the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is $E(G_1) \cup E(G_2)$. When $V(G_1) \cap V(G_2) = \varnothing$ we say that G_1 and G_2 are disjoint, and refer to $G_1 \cup G_2$ as the disjoint union of G_1 and G_2. For two graphs G_1 and G_2, the Cartesian product $G_1 \,\square\, G_2$ is a graph with vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, u_2)(v_1, v_2) : u_i, v_i \in V(G_i)$ for $i \in \{1, 2\}$ and either $u_1 = v_1$ and $u_2 \sim v_2$, or $u_2 = v_2$ and $u_1 \sim v_1\}$. If U_1 is a v_0, v_ℓ-path and U_2 is a v_ℓ, v_m-path, where U_1 and U_2 have only one vertex in common, namely v_ℓ, then the concatenation of U_1 and U_2 is the v_0, v_m-path $U_1 \circ U_2 = v_0v_1 \ldots v_\ell v_{\ell+1} \ldots v_m$. A hypercube of dimension n, denoted Q_n, is the graph formed by labeling a vertex with each of the 2^n binary sequences of length n, and joining two vertices with an edge if and only if their sequences differ in exactly one position.