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Counting arithmetical structures on paths and cycles

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ABSTRACT

Let G be a finite, connected graph. An arithmetical structure on G is a pair of positive integer vectors $\mathbf{d}, \mathbf{r}$ such that $(\text{diag}(\mathbf{d}) - A)\mathbf{r} = 0$, where A is the adjacency matrix of G. We investigate the combinatorics of arithmetical structures on path and cycle graphs, as well as the associated critical groups (the torsion part of the cokernels of the matrices $(\text{diag}(\mathbf{d}) - A)$). For paths, we prove that arithmetical structures are enumerated by the Catalan numbers, and we obtain refined enumeration results related to ballot sequences. For cycles, we prove that arithmetical structures are enumerated by the binomial coefficients $\binom{2n-1}{n-1}$, and we obtain refined enumeration results related to multisets. In addition, we determine the critical groups for all arithmetical structures on paths and cycles.

1. Introduction

This paper is about the combinatorics of arithmetical structures on path and cycle graphs. We begin by recalling some basic facts about graphs, Laplacians, and critical groups.

Let G be a finite, connected graph with $n \geq 2$ vertices, let A be its adjacency matrix, and let D be the diagonal matrix of vertex degrees. The *Laplacian matrix* $L = D - A$ has rank $n - 1$, with nullspace spanned by the all-ones vector $\mathbf{1}$. If we regard L as a $\mathbb{Z}$-linear transformation $\mathbb{Z}^n \to \mathbb{Z}^n$, the cokernel $\mathbb{Z}^n/\text{im}\, L$ has the form $\mathbb{Z} \oplus K(G)$; here $K(G)$, the *critical group* is finite abelian, with cardinality equal to the number of spanning trees of G, by the Matrix-Tree Theorem. The critical group is also known as the *sandpile group* or the *Jacobian*. The elements of the critical group represent long-term behaviors of the well-studied *abelian sandpile model* on G; see, e.g., [2,8,11].

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More generally, an *arithmetical structure* on G is a pair $(\mathbf{d}, \mathbf{r})$ of positive integer vectors such that $\mathbf{r}$ is primitive (the gcd of its coefficients is 1) and

$$(\operatorname{diag}(\mathbf{d}) - A)\mathbf{r} = \mathbf{0}.$$

This definition generalizes the Laplacian arithmetical structure just described, where $\mathbf{d}$ is the vector of vertex degrees and $\mathbf{r} = \mathbf{1}$. Note that each of $\mathbf{d}$ and $\mathbf{r}$ determines the other uniquely, so we may regard any of $\mathbf{d}$, $\mathbf{r}$, or the pair $(\mathbf{d}, \mathbf{r})$ as an arithmetical structure on G. Where appropriate, we will use the terms *arithmetical d-structure* and *arithmetical r-structure* to avoid ambiguity. The set of all arithmetical structures on G is denoted $\operatorname{Arith}(G)$, and the data G, $\mathbf{d}$, $\mathbf{r}$ together determine an *arithmetical graph*. As in the classical case, the matrix $L(G, \mathbf{d}) = \operatorname{diag}(\mathbf{d}) - A$ has rank $n - 1$ [12, Proposition 1.1]. The torsion part of coker L is the *critical group* of the arithmetical graph.

Arithmetical graphs were introduced by Lorenzini in [12] to model degenerations of curves. Specifically, the vertices of G represent components of a degeneration of a given curve, edges represent intersections of components, and the entries of $\mathbf{d}$ are self-intersection numbers. The critical group is then the group of components of the Néron model of the Jacobian of the generic curve (an observation attributed by Lorenzini to Raynaud). In this paper, we will not consider the geometric motivation, but instead study arithmetical graphs from a purely combinatorial point of view.

It is known [12, Lemma 1.6] that $\operatorname{Arith}(G)$ is finite for all connected graphs G. The proof of this fact is non-constructive (by reduction to Dickson's lemma), raising the question of enumerating arithmetical structures for a particular graph or family of graphs. We will see that when G is a path or a cycle, the enumeration of arithmetical structures on G is controlled by the combinatorics of Catalan numbers. In brief, the path $\mathcal{P}_n$ and the cycle $\mathcal{C}_n$ on n vertices satisfy

$$|\operatorname{Arith}(\mathcal{P}_n)| = C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}, \qquad |\operatorname{Arith}(\mathcal{C}_n)| = \binom{2n-1}{n-1} = (2n-1)C_{n-1}$$

(Theorems 3 and 30, respectively). These results were announced in [7].

We will refine these results, and show for example that the number of $\mathbf{d}$-structures on $\mathcal{P}_n$ with one prescribed d_i entry are given by the *ballot numbers*, a well-known combinatorial refinement of the Catalan numbers first investigated by Carlitz [5]. For cycles, we get a similar result where the ballot numbers are replaced by binomial coefficients. The critical group of an arithmetical structure on a path is always trivial, while for a cycle it is always cyclic of order equal to the number of occurrences of 1 in the associated arithmetical r-structure. Our approaches for these two families are similar: ballot sequences yield information about arithmetical structures for paths, while multisets produce information in the case of cycles. Our main results for paths and cycles mirror each other, as do the proof techniques we use.

Two graph operations that play a central role in our work are *subdivision* (or *blowup*) and *smoothing*. On the level of graphs, subdividing an edge inserts a new degree-2 vertex between its endpoints, while smoothing a vertex of degree 2 removes the vertex and replaces its two incident edges with a single edge between the adjacent vertices. These operations extend to arithmetical structures and preserve the critical group, as shown by Corrales and Valencia [7, Thms. 5.1, 5.3, 6.5], following Lorenzini [12, pp. 484–485]. These operations turn out to be key in enumerating arithmetical structures. Note that paths and cycles are special because they are precisely the connected graphs of maximum degree 2, hence can be obtained from very small graphs by repeated subdivision.

Looking ahead, many open questions remain about arithmetical graphs. It is natural to ask to what extent the arithmetical critical group $K(G, \mathbf{d}, \mathbf{r})$ behaves like the standard critical group. The matrix $L(G, \mathbf{d})$ is an *M-matrix* in the sense of numerical analysis (see, e.g., Plemmons [13]), so it admits a generalized version of chip-firing as described by Guzmán and Klivans [10]. One could also look for an analogue of the matrix-tree theorem, asserting that the cardinality of the critical group enumerates some tree-like structures on the corresponding arithmetical graphs, or for a version of Dhar's burning algorithm [8] that gives a bijection between those structures and objects like parking functions.

Enumerating arithmetical structures for graphs other than paths and cycles appears to be more difficult. For example, the arithmetical d-structures on the star $K_{n,1}$ can be shown to be the positive integer solutions to the equation

$$d_0 = \sum_{i=1}^{n} \frac{1}{d_i}.$$

A solution to this Diophantine equation is often called an *Egyptian fraction representation* of d_0. The numbers of solutions for $n \leq 8$ are given by sequence A280517 in [16]. A related problem, with the additional constraints $d_0 = 1$ and $d_1 \leq \cdots \leq d_n$, was studied by Sándor [14], who gave upper and lower bounds for the number of solutions; the upper bound was subsequently improved by Browning and Elsholtz [4]. The lower and upper bounds are far apart, and it is unclear even what asymptotic growth to expect.

2. Paths

We have two main goals in this section. First, we show in Theorem 9 that using r-structures one can partition the arithmetical structures on a fixed path into sets with cardinality given by ballot numbers, generalizing Theorem 3 which states that the total number of arithmetical structures is given by a Catalan number. Second, we show in Theorem 17 that using d-structures one can produce additional partitions of the arithmetical structures of a fixed path that again has distribution given by ballot numbers. We begin with some basic results about arithmetical structures on paths.