Discrete Mathematics 341 (2018) 2590–2601

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Gray codes for signed involutions

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ARTICLE INFO

Article history:
Received 4 May 2018
Received in revised form 17 May 2018
Accepted 4 June 2018

Keywords:
Gray code
Signed involution
CAT algorithm

ABSTRACT

In this paper we present two cyclic Gray codes for signed involutions. The first one has a natural construction, implemented by a CAT algorithm, based in the recursive formula for the number of signed involutions. The second code, although with a higher computational cost, has the smallest possible Hamming distance for this family of objects.

1. Introduction

The exhaustive generation and listing of a combinatorial class of objects according to a fixed parameter is one of the most important aims in combinatorics, with applications in a vast range of areas ranging from computer science and hardware or software testing, thermodynamic, biology and biochemistry [6,11]. A common approach is to list the objects such that successive objects differ by a well-defined closeness condition. These lists are usually called Gray codes, a term taken from Frank Gray [4], who patented the Binary Reflected Gray Code, $BRGC_n$, a list of all 2^n binary strings of length n in which successive strings differ by a single bit. We adopt the point of view of Walsh [16], defining a Gray code for a class of objects as an infinite set of word-lists with unbounded word-length such that the Hamming distance between any two adjacent words, in any list, is bounded independently of the word-length. The Hamming distance between any two adjacent words is the number of positions in which these two words differ. When this distance is preserved between the last and the first words, the code is said to be cyclic.

A great interest has been shown in the generation of Gray codes for permutations [5,13] and their restrictions or generalizations, such as permutations with a fixed number of cycles [1], derangement [2], involutions and fixed-point free involutions [15], multiset permutations [14], and signed permutations [7]. We present here two cyclic Gray codes for the set of n-length signed involutions [12].

The first code is constructed inductively using a natural construction derived from a formula for the number of signed involutions. In particular, our implementation of this code is able to generate all Gray codes for k-length signed involutions, for $k \leq n$. The second code is constructed by levels, each corresponding to involutions having a fixed number of disjoint transpositions, applying a variation of the Binary Reflected Gray Code to generate the elements of each level.

In the first code, each involution is transformed into its successor by a transposition, a rotation of three elements and/or one sign change, while in the second one this difference is reduced to a transposition or at most a pair of sign changes. This implies that the Hamming distance for this second code is 2, which is proven to be minimal (see Corollary 4.8). Although optimal in terms of distance between consecutive words, this second code has the disadvantage of being more elaborated and with a higher computational cost in comparison with the previous one, which has a more natural construction and is generated by a constant amortized time (CAT) algorithm. This computational comparison was numerically confirmed for n up to 15.

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https://doi.org/10.1016/j.disc.2018.06.011

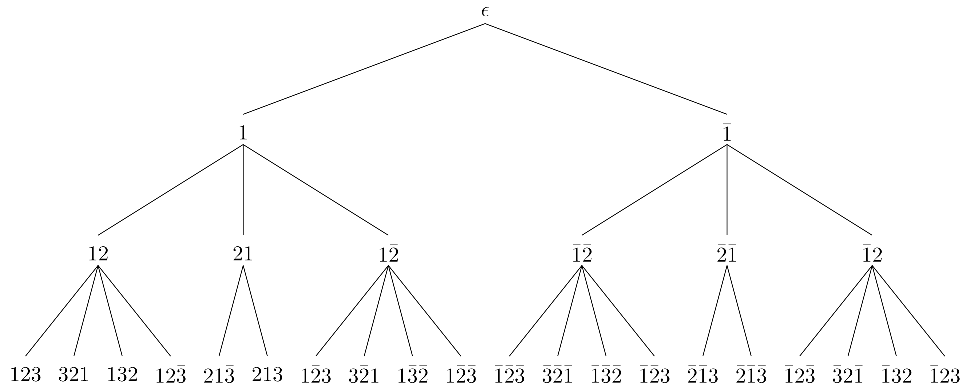

Fig. 3.1. Gray codes for $\mathfrak{I}_n$, for $n \leq 3$.

2. Notation and definitions

A *signed permutation* σ of length n is a permutation on the set $[\pm n] = \{\pm 1, \ldots, \pm n\}$ satisfying

$$\sigma(-i) = -\sigma(i). \tag{2.1}$$

Under the ordinary composition of mappings, the signed permutations of $[\pm n]$ form a group, called the hyperoctahedral group of rank n, denoted by $\mathfrak{S}_n^B$. The group of ordinary permutations is denoted by $\mathfrak{S}_n$. Eq. (2.1) indicates that the signed permutation σ is entirely defined by its values on $[n] = \{1, \ldots, n\}$. Therefore, we shall represent the elements $\sigma \in \mathfrak{S}_n^B$ in one line notation $\sigma = \sigma_1 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$ for $i \in [n]$. It also follows that $\mathfrak{S}_n^B$ has order $2^n n!$.

Given $i \in [\pm n]$, we define the sign function $sgn(i) = 0$ if $i > 0$, and $sgn(i) = 1$ otherwise. A signed permutation σ may also be represented by a pair (p, g), where $p \in \mathfrak{S}_n$ is the list of the numbers in σ without signs, and $g \in B_n$, the set of binary words of length n, is the list of the signs in σ, that is $p_i = |\sigma_i|$ and $g_i = sgn(\sigma_i)$ for every $i \in [n]$. For example, $\sigma = \bar{4}\,3\,2\,\bar{1}\,\bar{5} \in \mathfrak{S}_5^B$ corresponds to the pair $(43215, 10011) \in \mathfrak{S}_5 \times B_5$, where for ease of notation, we denote by $\bar{i}$ the negative element $-i$. Depending on the context, it will be clear which representation of signed permutations we are using.

A signed permutation $\sigma \in \mathfrak{S}_n^B$ is an *involution* if $\sigma^2 = id$, the identity element of $\mathfrak{S}_n^B$, and we denote by $\mathfrak{I}_n$ the set of all involutions in $\mathfrak{S}_n^B$. Note that if $\sigma \equiv (p, g)$ is an involution in $\mathfrak{S}_n^B$, then p is also an involution in $\mathfrak{S}_n$. The cycle decomposition of $\sigma \equiv (p, g) \in \mathfrak{I}_n$ is obtained by writing p as the disjoint union of transpositions and position fixed points, *i.e.* integers i for which $p_i = i$, and then associating the respective signs. For instance, for $\sigma = \bar{4}\,3\,2\,\bar{1}\,\bar{5} \equiv (43215, 10011)$, we have $43215 = (14)(23)(5) \in \mathfrak{S}_5$, and thus $\sigma = (\bar{1}\,\bar{4})(23)(\bar{5}) \in \mathfrak{S}_5^B$. When writing a transposition $(a\,b)$ we adopt the convention that $|a| < |b|$. The integers a and b are called, respectively, the *opener* and *closure* of $(a\,b)$ and, as part of an involution, must have the same sign. The common sign of both elements in a transposition is called a *paired sign*.

3. A CAT Gray code

We start this section giving a brief description of our construction. Having a Gray code for $\mathfrak{I}_{n-1}$, the main idea of our first algorithm is the following. Replace each involution w in the code for $\mathfrak{I}_{n-1}$ by a sequence of involutions of $\mathfrak{I}_n$, where the first and the last elements are obtained from w by adding a position fixed letter n, once signed and once unsigned. Each other element in the sequence, if any, is obtained from w by transposing a position fixed letter (see Definition 3.1) with the extra letter n or $\bar{n}$. Fig. 3.1 shows this construction for $n \leq 3$, where each row (reading from left to right) is a Gray code for $\mathfrak{I}_n$.

In the following, we show that this construction runs over all involutions in $\mathfrak{I}_n$.

Definition 3.1. An involution $\sigma \in \mathfrak{I}_n$ is said to be position fixed on the letter $j \in [n]$ if $|\sigma_j| = j$. Let F_n be the set of all involutions having at least one position fixed letter, and let F_n^i be the set of all position fixed involutions on the letter i, that is,

$$F_n^i = \{\sigma \in \mathfrak{I}_n : |\sigma_i| = i\} \text{ and } F_n = \cup_{i=1}^n F_n^i.$$

Finally, let $\mathfrak{I}_n' := \mathfrak{I}_n \setminus F_n^n$ be the set of involutions which are not position fixed on the letter n.

The next definition introduces the maps that formalize the construction of our algorithm.

Definition 3.2. Consider an integer $n \geq 2$.

(*a*) Let $\phi_n : \{\pm n\} \times \mathfrak{I}_{n-1} \to F_n^n$ be the map where $\phi_n(s, \sigma)$ is obtained from σ by adding the letter s on its rightmost position.