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On two unimodal descent polynomials

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ABSTRACT

The descent polynomials of separable permutations and derangements are both demonstrated to be unimodal. Moreover, we prove that the γ-coefficients of the first are positive with an interpretation parallel to the classical Eulerian polynomial, while the second is spiral, a property stronger than unimodality. Furthermore, we conjecture that they are both real-rooted.

1. Introduction

Many polynomials with combinatorial meanings have been shown to be unimodal; see the recent survey of Brändén [3]. Recall that a polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ of degree d is said to be *unimodal* if the coefficients are increasing and then decreasing, i.e., there is an index c such that $h_0 \le h_1 \le \cdots \le h_c \ge h_{c+1} \ge \cdots \ge h_d$. Let $p(t) = a_r t^r + a_{r+1} t^{r+1} + \cdots + a_s t^s$ be a real polynomial with $a_r \neq 0$ and $a_s \neq 0$. It is called *palindromic* (or *symmetric*) *of centre* $n/2$ if $n = r + s$ and $a_{r+i} = a_{s-i}$ for $0 \le i \le n/2$. For example, polynomials $1 + t$ and t are palindromic of centre $1/2$ and 1, respectively. Any palindromic polynomial $p(t) \in \mathbb{Z}[t]$ can be written uniquely [3,20] as

$$p(t) = \sum_{k=r}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k},$$

where $\gamma_k \in \mathbb{Z}$. If $\gamma_k \ge 0$ then we say that it is *γ-positive of centre* $n/2$. It is clear that the γ-positivity implies palindromicity and unimodality. Three prototypes of combinatorial γ-positive polynomials are the binomial polynomials $(1+x)^n$ with $n \in \mathbb{N}$, Eulerian polynomials and Narayana polynomials; see (1.1) and (1.2). For further γ-positivity results and problems, the reader is referred to the excellent exposition by Petersen [16] and the most recent survey by Athanasiadis [1]. The aim of this paper is to provide two new families of combinatorial unimodal polynomials, of which one is γ-positive (Theorem 1.1) and another is not palindromic but has spiral property, which also implies the unimodality (Theorem 1.2).

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. For a permutation $\pi \in \mathfrak{S}_n$, written as $\pi = \pi_1 \pi_2 \ldots \pi_n$, an index $i \in [n]$ is a *descent* (resp. *double descent*) of π if $\pi_i > \pi_{i+1}$ (resp. $\pi_{i-1} > \pi_i > \pi_{i+1}$), where $\pi_0 = \pi_{n+1} = +\infty$. Denote

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by $\mathrm{des}(\pi)$ and $\mathrm{dd}(\pi)$ the number of descents and double descents of π, respectively. It is known [8,16] (see also [15]) that the descent polynomial on $\mathfrak{S}_n$ is the nth *Eulerian polynomial*, which is γ-positive of centre $(n-1)/2$:

$$A_n(t) := \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}^{A} t^k (1+t)^{n-1-2k}, \tag{1.1}$$

where $\gamma_{n,k}^{A} = \#\{\pi \in \mathfrak{S}_n : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$.

Patterns in permutations have been extensively studied in the literature (see for instance Kitaev's book [13]). A permutation π is said to contain the permutation σ if there exists a subsequence of (not necessarily consecutive) entries of π that has the same relative order as σ, and in this case σ is said to be a pattern of π; otherwise, π is said to avoid σ. The set of permutations avoiding patterns $\sigma_1, \ldots, \sigma_r$ in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(\sigma_1, \ldots, \sigma_r)$. The descent polynomial over $\mathfrak{S}_n(231)$ is the nth *Narayana polynomial* [16, Chapter 2], which is also γ-positive of centre $(n-1)/2$; see [16, Theorem 4.2] or [17, Proposition 11.14] for an equivalent statement:

$$N_n(t) := \sum_{\pi\in\mathfrak{S}_n(231)} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}^{N} t^k (1+t)^{n-1-2k}, \tag{1.2}$$

where $\gamma_{n,k}^{N} = \#\{\pi \in \mathfrak{S}_n(231) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$.

A permutation avoiding patterns 2413 and 3142 is called a *separable permutation*. It is known (see [18,21]) that separable permutations are counted by the *large Schröder numbers*. The first few numbers are 1, 2, 6, 22, 90, 394, 1806, see oeis:A006318. Our first main result is the following γ-expansion for the descent polynomial on separable permutations.

Theorem 1.1. *We have*

$$S_n(t) := \sum_{\pi\in\mathfrak{S}_n(2413,3142)} t^{\mathrm{des}(\pi)} = \sum_{k\geq 0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k}^{S} t^k (1+t)^{n-1-2k}, \tag{1.3}$$

where $\gamma_{n,k}^{S} = \#\{\pi \in \mathfrak{S}_n(2413, 3142) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$. *Consequently, the polynomial* $S_n(t)$ *is* γ*-positive and a fortiori, palindromic and unimodal.*

For example, the first expansions of $S_n(t)$ read as follows:

$$S_1(t) = 1;\ \ S_2(t) = 1 + t;$$
$$S_3(t) = 1 + 4t + t^2 = (1 + t)^2 + 2t;$$
$$S_4(t) = 1 + 10t + 10t^2 + t^3 = (1 + t)^3 + 7t(1 + t);$$
$$S_5(t) = 1 + 20t + 48t^2 + 20t^3 + t^4 = (1 + t)^4 + 16t(1 + t)^2 + 10t^2;$$
$$S_6(t) = 1 + 35t + 161t^2 + 161t^3 + 35t^4 + t^5 = (1 + t)^5 + 30t(1 + t)^3 + 61t^2(1 + t).$$

The palindromicity $S_n(t) = t^{n-1}S_n(1/t)$ follows from the involution

$$\pi_1\pi_2\cdots\pi_n \mapsto \pi_n\pi_{n-1}\cdots\pi_1$$

and the fact that $\mathfrak{S}_n(2413, 3142)$ is invariant under this involution. Though this class of palindromic polynomials already exists in OEIS (see oeis:A175124), its interpretation as descent polynomials of separable permutations seems new. Note that both (1.1) and (1.2) can be proved using the *modified Foata–Strehl action* (see [2,9,15]) on $\mathfrak{S}_n$, but since $\mathfrak{S}_n(2413, 3142)$ is not invariant under this action, it is unclear how Theorem 1.1 could be deduced by the same manner.

A *derangement* is a fixed-point free permutation. Let $\mathfrak{D}_n$ be the set of derangements in $\mathfrak{S}_n$ and consider the descent polynomial of derangements

$$D_n(t) := \sum_{\pi\in\mathfrak{D}_n} t^{\mathrm{des}(\pi)}.$$

The first few values of $D_n(t)$ are listed as follows:

$$D_2(t) = t;\ \ D_3(t) = 2t;$$
$$D_4(t) = 4t + 4t^2 + t^3;$$
$$D_5(t) = 8t + 24t^2 + 12t^3;$$
$$D_6(t) = 16t + 104t^2 + 120t^3 + 24t^4 + t^5;$$
$$D_7(t) = 32t + 392t^2 + 896t^3 + 480t^4 + 54t^5.$$

We have the following spiral property for $D_n(t)$.