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A characterization of cycle-forced bipartite graphs

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ABSTRACT

A *forced cycle* C of a graph G is a cycle in G such that $G-V(C)$ has a unique perfect matching. A graph G is a *cycle-forced graph* if every cycle in G is a forced cycle. In this paper, we give a characterization of cycle-forced hamiltonian bipartite graphs.

1. Introduction

In this paper, we follow the notations and terminology in [1] except otherwise stated. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset M of $E(G)$ is called a *perfect matching* of G if no two edges in M are adjacent, and M covers all vertices of G. The *perfect-matching polytope* of a graph G is the convex hull of the incidence vectors of all perfect matchings of G. Chvátal [2] showed that two vertices of the prefect matching polytope are adjacent if and only if the symmetric difference of the two perfect matchings is a cycle. This result leads to the definition of the prefect matching graph of G. The *perfect matching graph* of G, denoted by $PM(G)$, is the graph whose vertices are the perfect matchings in G, with two vertices M_1 and M_2 adjacent in $PM(G)$ if the symmetric difference of M_1 and M_2 is a cycle, whose edges are alternate between M_1 and M_2. A graph G is *perfect matching compact* [4], or *PM-compact* for short, if the perfect matching graph of G is complete.

Wang, Shang, Lin, and Carvalho [5] characterized all PM-compact claw-free cubic graphs. Wang, Yuan, and Lin [6] characterized all PM-compact hamiltonian bipartite graphs. Wang, Lin, Carvalho, Lucchesi, Sanjith, and Little [4] characterized PM-compact bipartite and near bipartite graphs.

To present the known results about PM-compact bipartite graphs, we first give some definitions. A graph G is *matching-covered* if it is connected, and every edge of G is contained in a perfect matching. The graph K_2, possibly with the addition of multiple edges, is denoted by K_2^*. Let u be a vertex of degree 2 in G. The *retract* of G is the graph obtained from G by successively bicontracting vertices of degree 2 until either there are no vertices of degree 2 or at most two vertices remain, where to bicontract a vertex u of degree 2 is to contract both edges incident with u. A graph G is called an *outer partition* of $K_{p,q}$, if G is the resulting graph of the following two operations: (i) adding edges to nonadjacent vertices of $K_{p,q}$ to get a new graph G' under the condition that $G'[X]$ and $G'[Y]$ are paths, where (X, Y) is the bipartition of $K_{p,q}$, and (ii) replacing the edges of $G'[X]$ and $G'[Y]$ by paths of length 2 (See Fig. 1). Note that when $\min\{p, q\} \geq 2$, an outer partition of $K_{p,q}$ has exact two Hamilton cycle. We call G an *H-partition* of $K_{3,3}$ if $G \neq K_{3,3}$ and G can be obtained from $K_{3,3}$ by replacing the edges of a Hamilton cycle of $K_{3,3}$ by odd paths (See Fig. 2).

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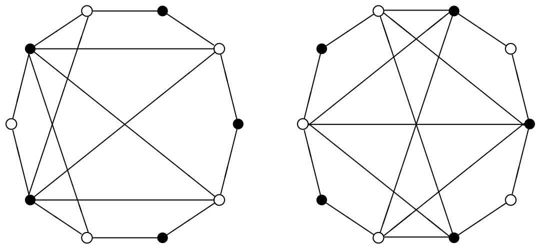

Fig. 1. Outer partition of $K_{2,4}$, $K_{3,3}$.

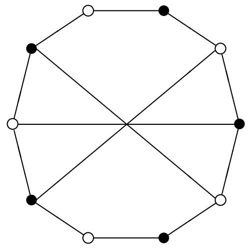

Fig. 2. H-partition of $K_{3,3}$.

Theorem 1.1 ([3]). *Every PM-compact bipartite graph G with $\delta(G) \geq 2$ is matching covered.*

Theorem 1.2 ([4]). *If G is a matching covered bipartite graph, then G is PM-compact if and only if the retract of G is $K_{3,3}$ or K_2^*.*

Theorem 1.3 ([6]). *Let G be a simple hamiltonian bipartite graph with $|V(G)| \geq 6$. Then G is PM-compact if and only if G is $K_{3,3}$, or G is a spanning hamiltonian subgraph of either an H-partition of $K_{3,3}$ or an outer partition of $K_{p,q}$, where $\min\{p, q\} \geq 2$.*

The definition of PM-compact graph implies that for a graph G having perfect matchings, G is PM-compact if and only if for each even cycle C of G, $G - V(C)$ has at most one perfect matching. A cycle C of a graph G is called a *forced cycle* if $G - V(C)$ has a unique perfect matching. A graph G is called a *cycle-forced graph* if every cycle in G is a forced cycle. Clearly, a cycle-forced graph is a PM-compact graph. By Theorem 1.1, a cycle-forced bipartite graph with $\delta(G) \geq 2$ is matching covered, and so is 2-connected. In this paper, we give a characterization of cycle-forced hamiltonian bipartite graphs, which is presented as follows.

An edge of G is called a 23-edge if one of its end vertices has degree 2 and the other one has degree at least 3, and a *22-edge* if both of its end vertices have degree 2. A bipartite graph G is a *maximal outerplanar bipartite graph* if for any edge e in the complement of G, either $G + e$ is nonbipartite or $G + e$ is not outerplanar. For a maximal outerplanar bipartite graph G with a planar embedding $\tilde{G}$ in which all vertices lie on the boundary of its outer face, G is connected, and if G is 2-connected, then the boundary of the outer face of $\tilde{G}$ is a Hamilton cycle, which is the only Hamilton cycle of G.

Let $\mathcal{B}$ represent the family of graphs satisfying the following two conditions:

(i) G is a 2-connected maximal outerplanar bipartite graph, and

(ii) if C is the Hamilton cycle of G, then $E(C)$ has two nonadjacent 22-edges, and all the other edges in $E(C)$ are 23-edges.

Theorem 1.4. *Let G be a simple hamiltonian bipartite graph with $|V(G)| \geq 6$. Then G is a cycle-forced graph if and only if G is $K_{3,3}$, an H-partition of $K_{3,3}$ or a spanning hamiltonian subgraph of a graph in $\mathcal{B}$.*

2. Preliminary results

In this section, we prove some preliminary results which will be used as basic tools to prove Theorem 1.4.

Lemma 2.1. *Let G be a cycle-forced graph, and let $e = uv$ be an edge of G. If H is a graph obtained from G by replacing the edge e by an odd path P, then H is a cycle-forced graph.*

Proof. Let C be a even cycle of H, and let $P = uv_1v_2 \cdots v_{2k}v$, $k \geq 1$. Then either $E(P) \cap E(C) = \emptyset$ or $E(P) \subseteq E(C)$. If $E(P) \cap E(C) = \emptyset$, then C is an even cycle of G, and so $G - V(C)$ has a unique perfect matching, denoted by M. If $uv \in M$, then $M \setminus \{uv\} \cup \{uv_1, v_2v_3, \ldots, v_{2k}v\}$ is the unique perfect matching of $H - V(C)$. If $uv \notin M$, then $M \cup \{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ is the unique perfect matching of $H - V(C)$. If $E(P) \subseteq E(C)$, then $C' = (C - V(P) \setminus \{u, v\}) + e$ is an even cycle of G. Note