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Perspective

Hamiltonian properties of polyhedra with few 3-cuts— A survey

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ABSTRACT

We give an overview of the most important techniques and results concerning the hamiltonian properties of planar 3-connected graphs with few 3-vertex-cuts. In this context, we also discuss planar triangulations and their decomposition trees. We observe an astonishing similarity between the hamiltonian behavior of planar triangulations and planar 3-connected graphs. In addition to surveying, (i) we give a unified approach to constructing non-traceable, non-hamiltonian, and non-hamiltonian-connected triangulations, and show that planar 3-connected graphs (ii) with at most one 3-vertex-cut are hamiltonian-connected, and (iii) with at most two 3-vertex-cuts are 1-hamiltonian, filling two gaps in the literature. Finally, we discuss open problems and conjectures.

1. Introduction

Hamiltonian cycles constitute a major branch of research in modern graph theory. Historically, such spanning cycles were already studied by Euler in 1759 in the closed variant of the “knight's tour” problem [22], which in non-mathematized form was treated by scholars in 9th century Baghdad and Kashmir. Applications of hamiltonicity range from combinatorial optimization and operations research [46] over coding theory [5], molecular chemistry [48] and fault-tolerance in networks [38] to music [1]. As far as we know, the hamiltonicity of polyhedra was first investigated in the 1850s, when Kirkman and Hamilton re-invented the concept independently [26,44] – for an excellent historical account, we refer the reader to [6]. Due to Steinitz's Theorem [61], we will here view polyhedra as 3-connected planar graphs, and *triangulations* shall be polyhedra in which every face is a triangle.

Investigating the hamiltonian properties of polyhedra gained popularity due to its connection with one of the most famous problems in all of mathematics: in 1884, Tait conjectured [62] that every cubic (i.e. 3-regular) polyhedron is hamiltonian, and had this conjecture been true, it would have implied the Four Color Theorem (which, then, was itself open). However, the conjecture turned out to be false and the first to construct a counterexample was Tutte in 1946, see [69]. The smallest counterexample is due to Noble Prize recipient Lederberg (and independently, Bosák and Barnette [37]) and has order 38. Deciding whether a given graph is hamiltonian is one of Karp's 21 NP-complete problems [42]. It remains NP-complete restricted to cubic polyhedra [27].

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Table 1
Hamiltonian properties of polyhedra with few 3-cuts. Green cells (marked ✓) indicate that every polyhedron with the specified number of 3-cuts satisfies the property, red cells (marked ✗) signify that there exist polyhedra which do not satisfy the property, and blank cells designate unknown behavior.

	Traceable	Hamiltonian	Hamiltonian-connected	2-edge-hamiltonian-connected	1-hamiltonian	2-hamiltonian
0	✓	✓(A,B)	✓(E)	✓(H)	✓(C)	✓(F)
1	✓	✓(D)	✓(K,L)	✗	✓	✗
2	✓	✓		✗	✓(L)	✗
3	✓	✓(G,J)		✗		✗
4	✓(J)		✗(K)	✗	✗	✗
5			✗	✗	✗	✗
6		✗(I)	✗	✗	✗	✗
7		✗	✗	✗	✗	✗
8	✗(J)	✗	✗	✗	✗	✗

A. *Whitney (1931) [73] (for triangulations)*
B. Tutte (1956) [70]
C. Nelson (1973) [52]
D. Thomassen (1978) [66]
E. Thomassen (1983) [68]
F. Thomas and Yu (1994) [64]
G. *Jackson and Yu (2002) [40] (for triangulations)*
H. Ozeki and Vrána (2014) [55]
I. Brinkmann, Souffriau, and Van Cleemput (2016) [10]
J. Brinkmann and Zamfirescu (2016) [11]
K. *Van Cleemput (2016) [72] (for triangulations)*
L. Ozeki, Van Cleemput, and Zamfirescu (this paper)

This survey will focus on the hamiltonian properties of polyhedra with few 3-cuts. (All cuts in this survey are vertex-cuts.) The first result in this line of research was obtained by Whitney in 1931: he showed that all 4-connected planar triangulations are hamiltonian [73]. In 1956, this was generalized by Tutte to all 4-connected polyhedra in his seminal work [70]. The works of Whitney and Tutte have seen many extensions; for instance to graphs of non-zero genus, both in the orientable and non-orientable case, graphs with small crossing number, as well as treating hamiltonian properties other than hamiltonicity itself, for instance traceability, hamiltonian-connectedness, or 1-hamiltonicity. Concerning these extensions, we refer the reader to the recent articles by Kawarabayashi and Ozeki [43], and Ozeki and Zamfirescu [56], and restrict ourselves to mentioning here a major conjecture of Grünbaum [29] and Nash-Williams [51]: *Every* 4-*connected toroidal graph is hamiltonian*. Since the result of Tutte, all of these theorems – be it in the planar or non-planar case – were proved using technical lemmas revolving around what are now known as Tutte-subgraphs and bridges. In this survey we shall concentrate on the planar case: in Table 1 we provide an overview of the current state of hamiltonicity and related concepts in polyhedra with few 3-cuts.

In Section 2 we introduce the basic terminology and concepts used throughout this paper. We will give an overview of all negative results from Table 1 in Section 3, while in Section 4 we introduce Tutte subgraphs and present the positive results from Table 1, as well as the tools used to prove them. In Section 5 we direct our attention to the subclass of triangulations and ascertain the changes with respect to the polyhedral case – for many readers, this will reveal a surprising insight: triangulations and polyhedra exhibit near-identical behavior concerning their hamiltonian properties, despite the fact that their sizes (i.e. the number of edges) may vary drastically for a fixed order. We finish in Section 6 by surveying open problems and partial results.

2. Basic terminology and concepts

All graphs considered are finite, simple, and undirected. If $G = (V(G), E(G))$ is a graph and $X \subseteq \binom{V(G)}{2}$, then we denote the graph $(V(G), E(G) \cup X)$ by $G \cup X$, and the graph $(V(G), E(G) \setminus X)$ by $G \setminus X$. If $Y \subset V(G)$, then $G - Y$ denotes the graph induced by G on the vertices $V(G) \setminus Y$, i.e. $G[V(G) \setminus Y]$. Abusing notation, we put $G - y = G - \{y\}$ for $y \in V(G)$. For an edge between vertices v and w we write vw. We say that a planar graph G is *plane* if we consider an embedding of G in the plane. Recall that by a theorem of Whitney polyhedra have unique planar embeddings [74].

A *hamiltonian cycle* (*hamiltonian path*) in a graph G is a cycle (path) which spans the vertex set $V(G)$. A graph is called *hamiltonian* if it contains a hamiltonian cycle, *traceable* if it contains a hamiltonian path, and *hamiltonian-connected* if there is a hamiltonian path connecting any two vertices in the graph. For a non-negative integer k, we say that a graph of order at least $k + 3$ is *k-hamiltonian* if the removal of any set of k vertices from the graph leaves a graph which is hamiltonian. We can consider hamiltonian as being 0-hamiltonian.

A *linear forest* is a forest in which each component is a path. For a non-negative integer k, we say that a graph G of order at least $k + 1$ is *k-edge-hamiltonian-connected* if for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G)\}$ with $|X| \leq k$ and the graph $(V(X), X)$ is a linear forest, $G \cup X$ has a hamiltonian cycle containing all edges of X. We can consider hamiltonian as being 0-edge-hamiltonian-connected, and hamiltonian-connected as being 1-edge-hamiltonian-connected. The concept was first introduced for $k = 2$ in [45], and mentioned in its general form in [55].