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Group-labeled light dual multinets in the projective plane[☆]

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ABSTRACT

In this paper we investigate light dual multinets labeled by a finite group in the projective plane $PG(2, \mathbb{K})$ defined over a field $\mathbb{K}$. We present two classes of new examples. Moreover, under some conditions on the characteristic of $\mathbb{K}$, we classify group-labeled light dual multinets with lines of length at least 9.

1. Introduction

Let $\mathrm{PG}(2, \mathbb{K})$ be the projective plane coordinatized by an algebraically closed field $\mathbb{K}$ of characteristic $p \geq 0$, and let Q be a finite set equipped with a binary operation $x \cdot y$. A possibility of linking the algebraic structure of Q to point-line incidences in $\mathrm{PG}(2, \mathbb{K})$ arises when Q may be embedded in $\mathrm{PG}(2, \mathbb{K})$ in the sense that there exist three maps from Q into $\mathrm{PG}(2, \mathbb{K})$, say $\alpha_1, \alpha_2, \alpha_3$, such that if $x \cdot y = z$ then $\alpha_1(x)$, $\alpha_2(y)$ and $\alpha_3(z)$ are three collinear points in $\mathrm{PG}(2, \mathbb{K})$. Here, the point-sets $\Lambda_i = \alpha_i(Q)$ with $i = 1, 2, 3$ are called components and assumed to be pairwise disjoint. What we can derive from the algebraic structure of Q in this way is a collection of properties of the incidence structure cut out on $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ by lines meeting each component. In most cases, such an incidence structure is rather involved and not very interesting, but there are significant exceptional cases, some of which arise from algebraic geometry.

In the simplest and perhaps nicest particular case, each line of $\mathrm{PG}(2, \mathbb{K})$ meeting two distinct components meets each component in an exactly one point; in particular $|\Lambda_1| = |\Lambda_2| = |\Lambda_3| = n$. Therefore, the incidence structure is a dual 3-net of order n, that is a 3-net of order n embedded in the dual plane of $PG(2, \mathbb{K})$. In this case, Q must be a quasigroup and the maps α_i are bijections. Conversely, if $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ is any dual 3-net in $\mathrm{PG}(2, \mathbb{K})$ then Λ is a quasigroup-labeled dual 3-net. In fact, there exists a quasigroup Q of order n together with a labeling of the sets $\Lambda_1, \Lambda_2, \Lambda_3$ by elements of Q such that points labeled by x, y, z are collinear if and only if $x \cdot y = z$ holds.

If Q is a group, which is an important particular case in our context, a fairly complete classification of all dual 3-nets is available for $p = 0$ or $p > n$; see [10]: Apart from four groups of smaller orders ($n = 8, 12, 24, 60$), a group-labeled dual 3-net is either algebraic, or of tetrahedron type. An algebraic dual 3-net has its components lying on a plane cubic Γ in

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$PG(2, \mathbb{K})$ and is labeled by an abelian group. More precisely, the group is cyclic when either Γ splits into three non-concurrent lines (triangle type dual 3-net), or into an irreducible conic and a line (conic–line type), otherwise the group is a subgroup of the Jacobian variety of Γ and is either cyclic or the direct product of two cyclic groups. A tetrahedron type dual 3-net is labeled by a dihedral group, and it can be viewed in the projective space $\mathrm{PG}(3, \mathbb{K})$ as the projection (from a point P on a plane $\pi = \mathrm{PG}(2, \mathbb{K})$) of six point sets of size n lying on the sides of a tetrahedron whenever these point sets, named λ_i, λ_i' with $i = 1, 2, 3$, have the following two properties: λ_i and λ_i' lie on opposite sides, and $\lambda_i \cup \lambda_i'$ are the components of a dual 3-net of order $2n$ in $\mathrm{PG}(3, \mathbb{K})$. In fact, if the center P of the projection is chosen outside the faces of the tetrahedron and $\Lambda_i = \alpha_i(\lambda_i) \cup \alpha_i(\lambda_i')$ for $i = 1, 2, 3$ then $(\Lambda_1, \Lambda_2, \Lambda_3)$ is a dual 3-net of order $2n$.

A more general, yet interesting case for applications occurs when Q is a quasigroup and each α_i is a bijection. The arising geometric configuration $(\Lambda_1, \Lambda_2, \Lambda_3)$ in $\mathrm{PG}(2, \mathbb{K})$, called *light dual multinet*, has still some regularity, such as $|\ell \cap \Lambda_1| = |\ell \cap \Lambda_2| = |\ell \cap \Lambda_3| = r$ for any line ℓ in $\mathrm{PG}(2, \mathbb{K})$ where r depends on ℓ and is called the length of ℓ. Obviously, any dual 3-net is a light dual multinet, but the converse is not true. A counterexample is obtained if the center of the projection in the above construction is chosen on a face (but a side) of the tetrahedron. The resulting light multinet of tetrahedron type is not a dual 3-net since some lines in $\mathrm{PG}(2, \mathbb{K})$ meet each component in n points; see [1,2]. If $(\Lambda_1, \Lambda_2, \Lambda_3)$ is a light dual multinet other than dual 3-nets, then some line in $\mathrm{PG}(2, \mathbb{K})$ meets each component in exactly $r > 1$ points. This shows that if $(\Lambda_1, \Lambda_2, \Lambda_3)$ is algebraic, then it is contained in a reducible plane cubic, and hence is either of pencil type, or triangular, or of conic–line type according as the cubic splits into three lines ℓ_1, ℓ_2, ℓ_3, either concurrent or not, or in an irreducible conic C plus a line ℓ.

In this paper, we present two new families of light dual multinets of order n. Those in the first family have order divisible by 3 and are of triangular type with ℓ_1, ℓ_2, ℓ_3 of length $n/3$. Light dual multinets in the second family have even order and they are of conic–line type with ℓ of length $n/2$.

As we pointed out in our papers [3,9], if $p > 0$ is small compared to the size of Q, many examples of dual 3-nets and light multinets arise from configurations of finite projective subplanes. Under the usual conditions on the characteristic of the underlying field, our main result describes group-labeled light dual multinets with a line of length at least 9.

Theorem 1.1. *Let* $\mathrm{PG}(2, \mathbb{K})$ *be the projective plane coordinatized by an algebraically closed field* $\mathbb{K}$ *of characteristic* $p \geq 0$*. Let* G *be a group of order* n *and assume* $p = 0$ *or* $p > n$*. Let* $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ *be a light dual multinet in* $\mathrm{PG}(2, \mathbb{K})$*, labeled by* G*. If* Λ *has a line of length at least* 9*, then exactly one of the following cases occurs:*

(i) Λ *is contained in a line.*
(ii) Λ *has a line of length* 3*.*
(iii) Λ *is either triangular, or of tetrahedron type, or of conic–line type.*

The paper is organized as follows. In Section 2 we put down the details on the labeling of light dual multinets and recall some facts of dual nets. In Section 3 we present our new examples of light dual multinets. In Section 4 we study light dual multinets labeled by the cyclic group. In Proposition 4.5 we show that such multinets are algebraic. In Section 5 we prove a series of lemmas on the non-extendability of the known light dual multinet constructions. Finally, in Section 6 we prove Theorem 1.1 using the geometric results obtained so far, and some theorems of Herstein and Horoševskiĭ on abstract finite groups.

For a thorough discussion on the applications of finite multinets in algebraic geometry, especially in the study of completely reducible fibers of pencils of hypersurfaces, and also in complex line arrangements and resonance theory; see [3–5,9–14].

2. Notation, terminology and background on dual multinets

2.1. Light dual multinets

A *quasigroup* is a set Q endowed with a binary operation $x \cdot y$ such that the equation $x \cdot y = z$ can be uniquely resolved whenever two of the three values $x, y, z \in Q$ are known. One denotes the solutions by left and right division $y = x \setminus z$ and $x = z/y$. Quasigroups with a multiplicative unit element are called *loops*. Two quasigroups $Q(\cdot)$ and $R(\circ)$ are *isotopic* if $\alpha(x) \circ \beta(y) = \gamma(x \cdot y)$ holds for some bijective maps $\alpha, \beta, \gamma : Q \to R$. If $Q(\cdot)$ and $R(\circ)$ are isotopic groups then they are isomorphic.

Definition 2.1. Let $\mathbb{K}$ be an algebraically closed field and $(Q, \cdot)$ be a finite quasigroup of order n. A *dual multinet labeled by* Q is a triple $(\alpha_1, \alpha_2, \alpha_3)$ of maps $Q \to \mathrm{PG}(2, \mathbb{K})$ such that for any $x, y \in Q$, the points $\alpha_1(x)$, $\alpha_2(y)$ and $\alpha_3(xy)$ are collinear. The dual multinet is said to be *light* if the maps $\alpha_1, \alpha_2, \alpha_3$ are injective.

If it is also requested that for any $x, y \in Q$, the points $\alpha_1(x)$, $\alpha_2(y)$ and $\alpha_3(z)$ are only collinear when $z = x \cdot y$ then the light dual multinet is a dual 3-net.

The point sets $\Lambda_i = \alpha_i(Q)$ are the *components* of the dual multinet and we view the map α_i as a *labeling* of the component Λ_i. However, if a dual multinet is not a dual 3-net it may happen that $(\Lambda_1, \Lambda_2, \Lambda_3)$ depends on the labelings. Therefore, $(\Lambda_1, \Lambda_2, \Lambda_3)$ stands for the dual multinet arising from Q whenever the labelings are clear from the context. Bearing this in