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# Cycles with a chord in dense graphs

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ABSTRACT

A cycle of order $k$ is called a $k$-cycle. A non-induced cycle is called a chorded cycle. Let $n$ be an integer with $n \geq 4$. Then a graph $G$ of order $n$ is chorded pancyclic if $G$ contains a chorded $k$-cycle for every integer $k$ with $4 \leq k \leq n$. Cream, Gould and Hirohata (Australas. J. Combin. 67 (2017), 463–469) proved that a graph of order $n$ satisfying $\deg_G u + \deg_G v \geq n$ for every pair of nonadjacent vertices $u, v$ in $G$ is chorded pancyclic unless $G$ is either $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$, the Cartesian product of $K_3$ and $K_2$. They also conjectured that if $G$ is Hamiltonian, we can replace the degree sum condition with the weaker density condition $|E(G)| \geq \frac{1}{4}n^2$ and still guarantee the same conclusion. In this paper, we prove this conjecture by showing that if a graph $G$ of order $n$ with $|E(G)| \geq \frac{1}{4}n^2$ contains a $k$-cycle, then $G$ contains a chorded $k$-cycle, unless $k = 4$ and $G$ is either $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$. Then observing that $K_{\frac{n}{2},\frac{n}{2}}$ and $K_3 \,\square\, K_2$ are exceptions only for $k = 4$, we further relax the density condition for sufficiently large $k$.



## 1. Introduction

In this paper, we only consider finite simple graphs. A *k-cycle* is a cycle of order $k$. A graph $G$ of order $n \geq 3$ is *pancyclic* if $G$ contains a $k$-cycle for every $k$ with $3 \leq k \leq n$.

To determine whether a given graph $G$ is Hamiltonian is an NP-complete problem. Therefore, there is little hope in obtaining a criterion for the existence of a Hamiltonian cycle which can be described in a polynomial-time algorithm. The hardness of the problem also affects sufficient conditions. Many sufficient conditions for the existence of a Hamiltonian cycle make a graph $G$ so dense that $G$ is not only Hamiltonian but it also satisfies stronger cycle properties. This situation is highlighted by Bondy's Meta-Conjecture.

**Bondy's Meta-Conjecture.** *Almost all sufficient conditions for the existence of a Hamiltonian cycle make a graph pancyclic, possibly with a small number of well-described families of exceptional graphs.*

Bondy's Meta-Conjecture has long served as a driving force in the research of cycles in graphs. Bondy [3] himself proved a result to support it. For a non-complete graph $G$, we let $\sigma(G)$ denote the minimum degree sum over all pairs of nonadjacent vertices in $G$. If $G$ is complete, we let $\sigma(G) = +\infty$. The classical Ore's Theorem states that every graph $G$ of order $n \geq 3$ with $\sigma(G) \geq n$ is Hamiltonian. Bondy proved that under the same hypothesis, $G$ is actually pancyclic unless $n$ is even and $G$ is $K_{\frac{n}{2},\frac{n}{2}}$. Moreover, he proved that once we have a Hamiltonian cycle, we no longer need the degree sum condition, but a simple density condition makes the graph pancyclic.

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**Theorem A** (*Bondy [3]*). *Every Hamiltonian graph G of order n with at least $\frac{1}{4}n^2$ edges is pancyclic unless n is even and G is $K_{\frac{n}{2},\frac{n}{2}}$. In particular, if $|E(G)| > \frac{1}{4}n^2$, then G is pancyclic.*

Theorem A has shed a new light on the distribution of cycle lengths. When we require a Hamiltonian cycle, the density condition of Theorem A is not strong enough. The union of $K_1$ and $K_{n-1}$ contains $\frac{1}{2}(n-1)(n-2)$ edges but it is not Hamiltonian. Even if we restrict ourselves to the class of $k$-connected graphs for some constant $k$, we still have $K_k \vee \left(kK_1 \cup K_{n-2k}\right)$, which is $k$-connected and contains $\frac{1}{2}n^2 - o(n^2)$ edges, but it is not Hamiltonian. However, once we have a Hamiltonian cycle in a graph $G$ of order $n$, $|E(G)| > \frac{1}{4}n^2$ guarantees the existence of cycles of all possible lengths.

Inspired by Theorem A, several studies have been conducted concerning the relationship between the density of a graph and cycles of a variety of lengths. Note that a bipartite graph is not pancyclic and since $K_{\frac{n}{2},\frac{n}{2}}$ appears as an exception in Theorem A, it is natural to restrict ourselves to non-bipartite graphs to further pursue this line of research. Then Häggkvist, Faudree and Schelp [8] relaxed the density condition.

**Theorem B** (*Häggkvist, Faudree and Schelp [8]*). *Every non-bipartite Hamiltonian graph of order n with more than $\frac{1}{4}(n-1)^2 + 1$ edges is pancyclic.*

Both Theorems A and B assume Hamiltonicity in the hypothesis. However, Brandt [4] proved that the existence of a Hamiltonian cycle is not related with the relationship between the density of a graph and the distribution of cycle lengths. Let $g(G)$ and $c(G)$ be the lengths of a shortest and a longest cycle in $G$, respectively.

**Theorem C** (*Brandt [4]*). *A non-bipartite graph G of order n with more than $(n-1)^2/4 + 1$ edges contains a k-cycle for every integer k with $3 \le k \le c(G)$.*

A graph $G$ is *weakly pancyclic* if $G$ contains a $k$-cycle for every integer $k$ with $g(G) \le k \le c(G)$. In the same paper, Brandt conjectured the density condition can be further relaxed if we consider weak pancyclicity.

**Conjecture 1** (*Brandt [4]*). *A non-bipartite graph G of order n with more than $(n-1)(n-3)/4 + 4$ edges is weakly pancyclic.*

Note that $(n-1)(n-3)/4 + 4 = n^2/4 - n + 19/4$. Bollobás and Thomason [2] gave a partial answer to this conjecture.

**Theorem D** (*Bollobás and Thomason [2]*). *A non-bipartite graph G of order n with at least $\lfloor n^2/4 \rfloor - n + 59$ edges contains a k-cycle for every integer k with $4 \le k \le c(G)$.*

A *chord* of a cycle $C$ is an edge joining two non-consecutive vertices of $C$. If there exists a chord of $C$, we say that $C$ is a *chorded cycle*. A chorded cycle of order $k$ is called a *chorded k-cycle*. A chord is one of the main tools in the study of cycle length distribution. Intuitively speaking, chords in a cycle enrich the cycle space of a graph and raise the chance of finding a cycle of required length or property. Actually, the proofs in [3] locate a cycle of desired length by using chords in a Hamiltonian cycle. In this sense, it is worth studying the distribution of chords in a cycle. And the first step in this direction is to find a chorded cycle.

Cream, Gould and Hirohata [6] studied a degree sum condition for a graph to have chorded cycles of all possible lengths. A graph $G$ of order $n \ge 4$ is *chorded pancyclic* if $G$ contains a chorded $k$-cycle for every integer $k$ with $4 \le k \le n$. Cream et al. proved that a graph satisfying Ore's degree sum condition is not only pancyclic but also chorded pancyclic, except for a balanced complete bipartite graph, plus one more. For graphs $G$ and $H$, let $G \,\square\, H$ denote the Cartesian product of $G$ and $H$.

**Theorem E** (*Cream, Gould and Hirohata [6]*). *A graph of order $n \ge 4$ with $\sigma(G) \ge n$ is chorded pancyclic unless G is $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$.*

Cream et al. also conjectured that if a graph $G$ is Hamiltonian, we can replace the degree sum condition with the density condition $|E(G)| \ge \frac{1}{4}n^2$.

In this paper, we affirmatively answer the above conjecture and further clarify the relationship between the density of a graph and the existence of a chorded cycle of specified length. The following is the main theorem of this paper.

**Theorem 1.** *Let G be a graph of order n with $|E(G)| \ge \frac{1}{4}n^2$ and let k be a positive integer. If G contains a k-cycle, then it contains a chorded k-cycle unless $k = 4$ and G is either $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$.*

By combining this theorem with Theorem A, we affirmatively answer the conjecture of Cream et al.

**Corollary 2.** *A Hamiltonian graph G of order $n \ge 4$ with $|E(G)| \ge \frac{1}{4}n^2$ is chorded pancyclic unless G is $K_{\frac{n}{2},\frac{n}{2}}$ or $K_3 \,\square\, K_2$.*

Theorem 1 suggests that the existence of a chorded cycle in a dense graph is independent of Hamiltonicity and pancyclicity. In a dense graph, we can discuss the existence of a chorded cycle of a specified length from the existence of a cycle of the same length. In this setting, we may be able to obtain a refined density condition. For example, a bipartite graph does not have a chorded 4-cycle. Therefore, as long as we seek a chorded 4-cycle, it is difficult to improve the condition $|E(G)| \ge \frac{1}{4}n^2$ in Theorem 1. However, if $n \ge 6$, $K_{\frac{n}{2},\frac{n}{2}}$ contains a chorded 6-cycle. Also, when we require a chorded 5-cycle