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Defective 2-colorings of planar graphs without 4-cycles and 5-cycles

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ABSTRACT

A 2-coloring is a coloring of vertices of a graph with colors 1 and 2. Define $V_i := \{v \in V(G) : c(v) = i\}$ for $i = 1$ and2. We say that G is (d_1, d_2)*-colorable* if G has a 2-coloring such that V_i is an empty set or the induced subgraph $G[V_i]$ has the maximum degree at most d_i for $i = 1$ and2. Let G be a planar graph without 4-cycles and 5-cycles. We show that the problem to determine whether G is $(0, k)$-colorable is NP-complete for every positive integer k. Moreover, we construct non-$(1, k)$-colorable planar graphs without 4-cycles and 5-cycles for every positive integer k. In contrast, we prove that G is (d_1, d_2)-colorable where $(d_1, d_2) = (4, 4), (3, 5),$ and$(2, 9)$.

1. Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A *k-coloring c* is a function $c : V(G) \to \{1, \ldots, k\}$. A *proper k-coloring* is a k-coloring such that every pair of adjacent vertices receive different colors. Appel and Haken [1,2] proved the famous four color theorem stating that every planar graph has a proper 4-coloring. Grötzsch [11] showed that every planar graph without 3-cycles has a proper 3-coloring.

Define $V_i := \{v \in V(G) : c(v) = i\}$. We call c a $(d_1, d_2, \ldots, d_k)$*-coloring* if V_i is an empty set or the induced subgraph $G[V_i]$ has the maximum degree at most d_i for every $i \in \{1, \ldots, k\}$. A graph G is called $(d_1, d_2, \ldots, d_k)$*-colorable* if G admits a $(d_1, d_2, \ldots, d_k)$-coloring. Thus the four color theorem [1,2] can be restated as every planar graphs is $(0, 0, 0, 0)$-colorable. For improper 3-colorability of planar graph, Cowen, Cowen, and Woodall showed that every planar graph is $(2, 2, 2)$-colorable [9]. Eaton and Hull [10] proved that $(2, 2, 2)$-colorability is optimal by showing non-$(k, k, 1)$-colorable planar graphs for every k.

Let $\mathcal{F}$ denote the set of planar graphs without 4-cycles and 5-cycles. The famous Steinberg's conjecture proposes that every $G \in \mathcal{F}$ has a proper 3-coloring. Recently, this conjecture is disproved by Cohen-Addad, Hebdige, Král, Li, and Salgado [8]. One way to relax the conjecture is allowing some color classes to be improper. Xu, Miao, and Wang [14] proved that G is $(1, 1, 0)$-colorable, and Chen et al. [5] proved that G is $(2, 0, 0)$-colorable for every $G \in \mathcal{F}$.

A $(d_1, d_2, \ldots, d_k)$-coloring can be called a *defective k-coloring*. Many papers investigate defective 2-colorings of planar graphs in various settings. Montassier and Ochem [13] constructed planar graphs of girth 4 that are not (i, j)-colorable for every i, j. Borodin, Ivanova, Montassier, Ochem, and Raspaud [3] constructed planar graphs of girth 6 that are not $(0, k)$-colorable for every k. On the other hand, for every planar graph G of girth 5, Havet and Sereni [12] showed that G is $(2, 6)$-colorable and $(4, 4)$-colorable, and Choi and Raspaud [7] showed that G is $(3, 5)$-colorable.

In this paper, we obtain some results about defective 2-coloring of $G \in \mathcal{F}$ as follows. In Section 2, we show that the problem to determine whether $G \in \mathcal{F}$ is $(0, k)$-colorable is NP-complete for every positive integer k. In Section 3, we construct

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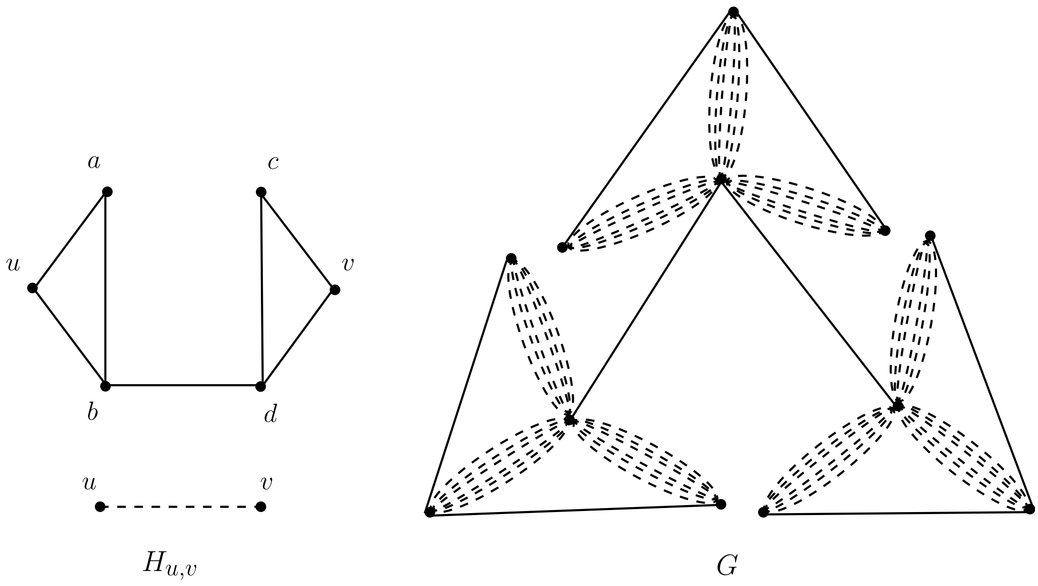

Fig. 1. A non-$(1, k)$-colorable planar graph G without 4-cycles and 5-cycles.

$G \in \mathcal{F}$ for a positive integer k such that G is non-$(1, k)$-colorable. Section 4 provides tools to show the contrast results as follows. We prove that $G \in \mathcal{F}$ is (d_1, d_2)-colorable where $(d_1, d_2) = (4, 4), (3, 5)$, and $(2, 9)$ in Sections 5–7, respectively. In Section 8, we summarize our results and related results. Some comments and open problems are also included.

Other definitions that we use in this work are as follows. A *k-vertex* (respectively, *k^+-vertex* and *k^--vertex*) is a vertex of degree k (respectively, at least k and at most k.) The same notation is applied for faces. A $(d_1, d_2, \ldots, d_k)$-*face* f is a face of degree k where all vertices on f have degree $d_1, d_2, \ldots, d_k$. If v is not on a 3-face f but v is adjacent to a 3-vertex u on f, then we call f a *pendant face* of a vertex v and v is a *pendant neighbor* of a vertex u on f. If a 3-face (respectively, 2-vertex) is incident to a 2-vertex (respectively, 3-face), then it is called a *bad* 3-*face* (respectively, *bad* 2-*vertex*). Otherwise, it is a *good* 3-*face* (respectively, *good* 2-*vertex*).

2. NP-completeness of $(0, k)$-colorings

Theorem 1 ([13]). *Let $g_{k,j}$ be the largest integer g such that there exists a planar graph of girth g that is not (k, j)-colorable. The problem to determine whether a planar graph with girth $g_{k,j}$ is (k, j)-colorable for $(k, j) \neq (0, 0)$ is NP-complete.*

Theorem 2. *The problem to determine whether $G \in \mathcal{F}$ is $(0, k)$-colorable is NP-complete for every positive integer k.*

Proof. We use a reduction from the problem in Theorem 1 to prove that $(0, k)$-coloring for $G \in \mathcal{F}$ is NP-complete. It can be obtained from [13] that $6 \leq g_{0,1} \leq 10$. Let G be a graph of girth $g_{0,1}$. Take $k - 1$ copies of 3-cycles $v_i v_i' v_i''$ $(i = 1, \ldots, k - 1)$ for every vertex v of G. The graph H_k is obtained from G by identifying v_i (in a 3-cycle $v_i v_i' v_i''$) to v for every vertex v. The resulting graph H_k has neither 4-cycles nor 5-cycles.

Suppose G has a $(0, 1)$-coloring c. If $c(v) = 1$, then we extend a coloring to $c(v_i') = c(v_i'') = 2$ for every $i = 1, \ldots, k - 1$. If $c(v) = 2$, then we extend a coloring to $c(v_i') = 1$ and $c(v_i'') = 2$ for every $i = 1, \ldots, k - 1$. One can see that c is a $(0, k)$-coloring of H_k. Suppose H_k has a $(0, k)$-coloring c. Consider $v \in V(G)$ with $c(v) = 2$. By construction, v has at least $k - 1$ neighbors in $V(H_k) - V(G)$ with color 2. Thus v has at most one neighbor with the same color 2 in $V(G)$. It follows that c with restriction to $V(G)$ is a $(0, 1)$-coloring of G. Hence G is $(0, 1)$-colorable if and only if H_k is $(0, k)$-colorable. This completes the proof. □

3. Non-$(1, k)$-colorable planar graphs without 4-cycles and 5-cycles

We construct a non-$(1, k)$-colorable planar graph G without 4-cycles and 5-cycles. Consider the graph $H_{u,v}$ shown in Fig. 1.

The vertices a, b, c, and d cannot receive the same color 1. Now, we construct the graph S_z as follows. Let z be a vertex and $x_1x_2x_3$ be a path. Take $2k + 1$ copies H_{u_i,v_j} of $H_{u,v}$ with $1 \leq i \leq 2k + 1$ and $1 \leq j \leq 3$. Identify every u_i with z and identify v_j with x_j. Finally, we obtain G from three copies S_{z_1}, S_{z_1}, and S_{z_3} by adding the edges z_1z_2 and z_2z_3. In every $(1, k)$-coloring of G, the path $z_1z_2z_3$ contains a vertex z with color 2. In the copy of S_z corresponding to z, the path $x_1x_2x_3$ contains a vertex x with color 2. Since every of z and x has at most k neighbors colored 2, one of $2k + 1$ copies of $H_{u,v}$ between z and x, does not contain a neighbor of z or a neighbor of x with color 2. This copy is not $(1, k)$-colorable, and thus G is not $(1, k)$-colorable.