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Minimum supports of eigenfunctions of Johnson graphs ☆

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ABSTRACT

We study the weights of eigenvectors of the Johnson graphs $J(n, w)$. For any $i \in \{1, \dots, w\}$ and sufficiently large $n, n \geq n(i, w)$ we show that an eigenvector of $J(n, w)$ with the eigenvalue $\lambda_i = (n - w - i)(w - i) - i$ has at least $2^i \binom{n-2i}{w-i}$ nonzeros and obtain a characterization of eigenvectors that attain the bound.

1. Introduction

Let $G = (V, E)$ be an undirected graph. A real-valued function $f : V \to R$ is called a λ-*eigenfunction* of G if the following equality holds for any $x \in V$:

$$\lambda f(x) = \sum_{y \in V : (x,y) \in E} f(y).$$

In other words, f is a λ-eigenfunction of G if its vector of values $\overline{f}$ is an eigenvector of the adjacency matrix A_G of G with eigenvalue λ or $\overline{f}$ is the all-zero vector, i.e. the following holds:

$$A_G \overline{f} = \lambda \overline{f}.$$

The vertices of the *Hamming graph* $H(n)$ are the binary vectors of length n, where two vectors are adjacent if they differ in exactly one coordinate position. *The Hamming distance* $d(x, y)$ between a pair of vectors x and y of length n is the Hamming graph distance between x and y, i.e. the number of positions at which the corresponding symbols are different. *The support* of a real-valued function (or vector) f denoted by *supp*(f) is the set of vertices where f is nonzero. By $f \equiv 0$ we mean that the support of f is empty. *The weight* $wt(x)$ of a vector x is the number of nonzero symbols of x.

The vertices of the Johnson graph $J(n, w)$ are the binary vectors of length n with w ones, where two vectors are adjacent if they have exactly $w - 1$ common ones. Note that the vertices of $J(n, w)$ are vertices of $H(n)$ of weight w, with the Johnson graph distance being the Hamming graph distance halved.

Various combinatorial objects with extreme characteristics could be defined in terms of eigenfunctions with certain restrictions. In particular, several important notions, such as $(w - 1) - (n, w, 1)$-designs (including Steiner triple and quadruple systems), equitable 2-cell partitions and perfect codes could be defined as eigenfunctions of the Johnson graphs [7, Chapter 4] and [15]. The symmetric difference of a pair of such objects (for example, Steiner triple systems) is a bitrade [12]. In case of the Johnson graphs, bitrades of small size play an important role in the classification and

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characterization problems (for example, see [2,12,17]) and proved to be a useful constructive tool for Steiner triple and quadruple systems [2,1]. Moreover, the topic of the current paper is related to the question of existence of 1-perfect codes in different graphs which is one of the most captivating problems in combinatorial coding theory. For $n \leq 2^{250}$ it is known that no such codes exist in the Johnson graphs $J(n, w)$, see [9]. The study of bitrades of 1-perfect codes may lead to an improvement of this problem. For another thing, there is a related problem of a distribution invariant of an association scheme asking the smallest number of non-negative values in a vector from a linear space. In case of the eigenspace corresponding to the largest nontrivial eigenvalue of the Johnson graph a conjecture for the solution of this problem was proposed in [14]. In this light, the question of finding the size of minimum support of nonzero eigenfunctions of the Johnson graphs for arbitrary fixed eigenvalues is tempting and intriguing.

For surveys on combinatorial objects connected with eigenfunctions, their bitrades and general theory the reader is referred to the works of Krotov et al. [12,11], Cho [4,5] and the book of Colbourn and Dinitz [6, Chapter 6, Section 60].

In the current paper the minimum support question for nonzero eigenfunctions of the Johnson graphs $J(n, w)$ with the eigenvalue $(w-i)(n-w-i)-i$ for any i, w and n, $n \geq n(i, w)$ is solved and a characterization of minimum support functions is obtained. In particular, we showed that the minimum size of the support is at least $2^i\binom{n-2i}{w-i}$ for n large enough. The solution for the problem in case of the minimum eigenvalue is 2^w [10] and it is attained on a class of so-called Steiner bitrades [10,12] that include Pasch-configuration.

2. Preliminaries

2.1. Induced eigenfunctions and eigenvalues of Johnson graphs

Let f be a real-valued function defined on the vertices of the Johnson graph $J(n, i)$. Define the function $I^{i,w}(f)$ on the vertices of $J(n, w)$ as follows:

$$I^{i,w}(f)(x) = \sum_{y, wt(y)=i, d(x,y)=|w-i|} f(y).$$

The function $I^{i,w}(f)$ is called *induced* in $J(n, w)$ by f [3]. The idea of using induced functions for representation of the Johnson scheme has been exploited since the beginning of its study, see for example [8]. In [3] the concept was generalized to a wider class of graphs. For the Johnson graphs the following result is well-known.

Theorem 1. *1. Let f be a λ-eigenfunction of $J(n, i)$. Then if $i \leq w$ then $I^{i,w}(f)$ is a $(\lambda + (w-i)(n-i-w))$-eigenfunction of $J(n, w)$.*
2. Let f be a real-valued function on the vertices of $J(n, w)$. Then $I^{w,w-1}(f) \equiv 0$ iff f is a $(-w)$-eigenfunction.

Proof. The sketch of the proof is done by induction on w. The second and the first statements of the theorem for $i = w-1$ are known, see the proof for example in [3, Theorem 1]. In general case it is easy to see that for any f we have: $(w-i)!I^{i,w}(f) = I^{w-1,w}(\dots(I^{i+1,i+2}(I^{i,i+1}(f))))$, which finishes the proof. □

Let $'$ be a bijection of two disjoint subsets M and M' of coordinate positions of size i. For a subset I of M denote the set of its images by I': $I' = \{m' : m \in I\} \subseteq M'$. Define the function $f^{i,w,n}$ on the vectors of weight w and length n in the following manner:

$$f^{i,w,n}(x) = (-1)^{|M \cap supp(x)|}, \text{ if } |supp(x) \cap (M \cup M')| = i \text{ and}$$

$$(supp(x) \cap M)' \cup (supp(x) \cap M') = M',$$

and $f^{i,w,n}(x) = 0$ otherwise. The main result of the current paper is that $f^{i,w,n}$ is the minimum support nonzero eigenfunction of the Johnson graphs $J(n, w)$ asymptotically on n.

Proposition 1. *The function $f^{i,w,n}$ is a $((w-i)(n-w-i)-i)$-eigenfunction of $J(n, w)$ with the support of size $2^i\binom{n-2i}{w-i}$.*

Proof.

The proof relies on Theorem 1. We show that $f^{i,i,n}$ is a $(-i)$-eigenfunction of the Johnson graph $J(n, i)$ and that $f^{i,w,n} = I^{i,w}(f^{i,i,n})$, where the functions $f^{i,w,n}$ and $f^{i,i,n}$ are obtained from the same pair of sets M and M' of sizes i.

Let x be a binary vector of length n and weight $i-1$. Consider the values of $I^{i,i-1}(f^{i,i,n})$ that could be expressed as follows:

$$I^{i,i-1}(f^{i,i,n})(x) = \sum_{y:wt(y)=i, supp(x) \subset supp(y)} f^{i,i,n}(y).$$

We have several cases. If $|(M \cap supp(x))' \cup (M \cap supp(x))| = i-1$ then there are just two elements $m \in M$ and $m' \in M'$ neither of which belongs to $supp(x)$. Therefore $I^{i,i-1}(f^{i,i,n})(x)$ is zero, since exactly two summands, $f^{i,i,n}(y)$ for y such that $supp(y) = supp(x) \cup \{m\}$ or $supp(y) = supp(x) \cup \{m'\}$ are equal to -1 and 1, while the other summands are zeros. In the