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Mock threshold graphs

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ABSTRACT

Mock threshold graphs are a simple generalization of threshold graphs that, like threshold graphs, are perfect graphs. Our main theorem is a characterization of mock threshold graphs by forbidden induced subgraphs. Other theorems characterize mock threshold graphs that are claw-free and that are line graphs. We also discuss relations with chordality and well-quasi-ordering as well as algorithmic aspects.

1. Introduction

We define, and study the surprisingly many properties of, a new class of graphs: a simple generalization of threshold graphs that we call "mock threshold graphs". One reason to study mock threshold graphs is that, like threshold graphs, they are perfect. We characterize the class of mock threshold graphs by forbidden induced subgraphs; we discuss their properties of chordality, planarity, claw-freeness, and well-quasi-ordering; and we find the line graphs that are mock threshold. We also treat algorithmic aspects of mock threshold graphs.

A graph G is said to be *threshold* if there are a function $w : V(G) \to \mathbb{R}$ and a real number t such that there is an edge between two distinct vertices u and v if and only if $w(u) + w(v) > t$ [3]. The class of threshold graphs has been studied in great detail, mainly because of its simple structure. There is an entire book about threshold graphs [13]. A fundamental theorem characterizes the class by forbidden induced subgraphs.

Theorem 1 (*[3]*)**.** *A graph is threshold if and only if it contains no induced subgraph isomorphic to* $2K_2$, P_4, *or* C_4.

Another fundamental fact about threshold graphs is their characterization by vertex ordering. If G is a graph and $X \subseteq V(G)$, then $G : X$ denotes the subgraph of G induced on X.

Theorem 2. *A graph G is threshold if and only if G has a vertex ordering $v_1, \dots, v_n$ such that for every i $(1 \le i \le n)$ the degree of v_i in $G : \{v_1, \dots, v_i\}$ is* 0 *or* $i - 1$.

By relaxing this characterization slightly we get a new, bigger class of graphs.

Definition 3. A graph G is said to be *mock threshold* if there is a vertex ordering $v_1, \dots, v_n$ such that for every i $(1 \le i \le n)$ the degree of v_i in $G : \{v_1, \dots, v_i\}$ is 0, 1, $i - 2$, or $i - 1$. We write $\mathcal{G}_{\mathrm{MT}}$ for the class of mock threshold graphs.

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We call such an ordering an *MT-ordering*. Note that a graph can have several MT-orderings. There are several easy but important consequences of the definition.

Although the class of mock threshold graphs is not closed under taking subgraphs, it is hereditary in the sense of induced subgraphs: Every induced subgraph of a mock threshold graph is mock threshold. Thus, as with threshold graphs, there exists a characterization by forbidden induced subgraphs, which we describe in Section 4 as our main theorem.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs H of G (Berge 1961). It is *chordal* if every induced cycle in it is a triangle [5]. It is *split* if its vertex set can be partitioned into a clique and a stable set [7]. It is *weakly chordal* if every induced cycle in it or its complement is either a triangle or a square [10]. We have the following chain of inclusions:

Threshold ⊂ Split ⊂ Chordal ⊂ Weakly Chordal ⊂ Perfect.

All the inclusions except the last one are easy. The last inclusion can be proved directly [10] or one can use the Strong Perfect Graph Theorem [2] to see the inclusion immediately.

Mock threshold graphs fit imperfectly into this chain of inclusions. They are perfect (Proposition 5) and indeed weakly chordal (Corollary 12) but not necessarily chordal. For instance, the cycle of length four is a mock threshold graph that is not chordal.

Proposition 4. *The complement of a mock threshold graph is also mock threshold.*

Proof. The same vertex ordering works. □

Proposition 5. *A mock threshold graph is perfect.*

Proof. Adding an isolated vertex or a leaf preserves perfection. The Weak Perfect Graph Theorem [12] tells us that a graph is perfect if and only if its complement is also perfect. Hence the four operations of constructing a mock threshold graph from K_1, viz., adding an isolated vertex, adding a leaf, adding a dominating vertex, and adding a vertex that dominates all but one vertex, preserve perfection. Hence every mock threshold graph is perfect. □

2. Preliminaries

All our graphs are finite and simple, that is, we allow neither loops nor multiple edges. When we say G contains H, we mean that G contains H as a subgraph. Let k be a positive integer. Then P_k, C_k, K_k denote the path, cycle, and complete graph, respectively, on k vertices. We denote the complement of G by $\overline{G}$. The neighborhood of a vertex v is denoted by $N(v)$. For a positive integer k, the *k-core* of a graph G is the graph obtained from G by repeatedly deleting vertices of degree less than k. It is routine to show that this is well defined.

For G a graph, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively. The chromatic number of G, denoted by $\chi(G)$, is the smallest positive integer k such that its vertices can be colored with k colors so that adjacent vertices receive different colors. The clique number of G, denoted by $\omega(G)$, is the largest positive integer k such that the complete graph on k vertices is a subgraph of G. The *codegree* $\overline{d}(v)$ of a vertex v in G is its degree in the complement $\overline{G}$.

3. Simple properties

We list some easy but useful properties and examples.

Lemma 6. *Let G be a graph on n vertices with $2 \le \delta(G) \le \Delta(G) \le n-3$. Then G is not mock threshold.*

A *removable vertex* is a vertex whose degree or codegree is at most 1. Thus, every mock threshold graph has a removable vertex.

Lemma 7. *Let G be a graph. Then G is mock threshold if and only if G has a removable vertex v and $G - \{v\}$ is mock threshold.*

Proof. If G is mock threshold, then every induced subgraph of G is also mock threshold. This proves one direction. For the other direction, assume that v is a removable vertex in G and that $G - \{v\}$ is mock threshold. Thus it has an MT-ordering. Add v as the last vertex to this MT-ordering; since v is removable, this is an MT-ordering for G. Hence G is mock threshold. □

K_n and $K_{2,n}$ are mock threshold. A forest is a mock threshold graph. It is easy to see that a graph is mock threshold if and only if its 2-core is mock threshold.

Lemma 8. *A graph consisting of two disjoint cycles is not mock threshold.*

Proof. The graph has no removable vertices. □