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Triangle-free graphs with no six-vertex induced path

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ABSTRACT

The graphs with no five-vertex induced path are still not understood. But in the triangle-free case, we can do this and one better; we give an explicit construction for all triangle-free graphs with no six-vertex induced path. Here are three examples: the 16-vertex Clebsch graph, the graph obtained from an 8-cycle by making opposite vertices adjacent, and the graph obtained from a complete bipartite graph by subdividing a perfect matching. We show that every connected triangle-free graph with no six-vertex induced path is an induced subgraph of one of these three (modulo some twinning and duplication).

1. Introduction

Graphs in this paper are without loops and multiple edges, and finite unless we say otherwise. If G is a graph and $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X; and we say that G *contains* H if some induced subgraph of G is isomorphic to H. If G has no induced subgraph isomorphic to H, we say G is H*-free*, and if $\mathcal{C}$ is a set of graphs, G is $\mathcal{C}$*-free* if G is H-free for all $H \in \mathcal{C}$. We denote the k-vertex path graph by P_k. Our objective in this paper is to find a construction for all $\{P_6, K_3\}$-free graphs.

Constructing all P_5-free graphs remains open, although it has been heavily investigated, mostly because P_5 is one of the minimal graphs for which the Erdős–Hajnal conjecture [2,3] is unsolved. It is also not clear whether we know how to construct all K_3-free graphs (it depends on what exactly counts as a "construction"). But we can construct all $\{P_5, K_3\}$-free graphs, and indeed all $\{P_6, K_3\}$-free graphs, and the answer is surprisingly pretty.

The Clebsch graph, shown in Fig. 1, is the most interesting of the $\{P_6, K_3\}$-free graphs. Here are three alternative definitions for it.

- Take the five-dimensional cube graph, and identify all pairs of opposite vertices.
- Take the elements of the field $GF(16)$, and say two of them are adjacent if their difference is a cube.
- Start with the Petersen graph; for each stable subset X of cardinality four (there are five such subsets) add a new vertex adjacent to the vertices in X; and then add one more vertex adjacent to the five new vertices. (This third definition is the least symmetric but in practice we found it the most helpful).

For every edge uv of the Clebsch graph, the subgraph induced on the set of vertices nonadjacent to u, v is a three-edge matching, and from this it follows that the graph is P_6-free. We say G is *Clebschian* if G is contained in the Clebsch graph.

There is another kind of $\{P_6, K_3\}$-free graph that we need to discuss, the following. Take a complete bipartite graph $K_{n,n}$ with bipartition $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$, and subdivide each edge a_ib_i; that is, for $1 \le i \le n$ we delete the edge a_ib_i, and add a new vertex c_i adjacent to a_i, b_i. This graph, H say, is $\{P_6, K_3\}$-free, and we say a graph G is *climbable* if it is isomorphic to an induced subgraph of H for some n (see Fig. 2).

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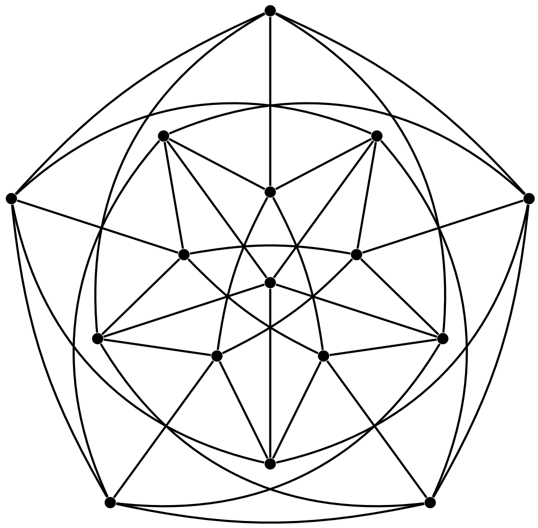

Fig. 1. The Clebsch graph.

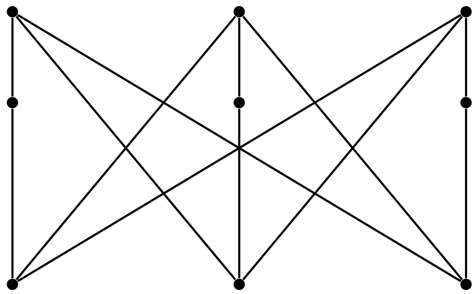

Fig. 2. A climbable graph.

Our aim is to prove something like "every $\{P_6, K_3\}$-free G is either climbable or Clebschian", but by itself this is not true. For instance, we have to assume G is connected, because otherwise the disjoint union of two Clebsch graphs would be a counterexample. But just assuming connectivity is not enough. For instance, if v is a vertex of a $\{P_6, K_3\}$-free graph, then we could add a new vertex with the same neighbours as v, and the enlarged graph would still be $\{P_6, K_3\}$-free. Let us say two vertices are *twins* if they are nonadjacent and have the same neighbour sets. Thus we need to assume that G has no twins. There are two other "thickening" operations of this kind that we define later.

Our main result is the following (although some definitions have not yet been given):

1.1. *Let G be a connected $\{P_6, K_3\}$-free graph without twins. Then either*

- *G is Clebschian, climbable, or a V_8-expansion; or*
- *G admits a nontrivial simplicial homogeneous pair.*

There has already been work on this and similar questions:

- In [7], Randerath, Schiermeyer and Tewes proved that every connected $\{P_6, K_3\}$-free graph which is not 3-colourable, and in which no vertex dominates another, is an induced subgraph of the Clebsch graph.
- Brandstädt, Klembt and Mahfud [1] proved that every $\{P_6, K_3\}$-free graph can be decomposed in a certain way that implies that all such graphs have bounded clique-width.
- Lozin [6] gave a construction for all bipartite graphs not containing what he called a "skew star", obtained from P_6 by adding one more vertex adjacent to its third vertex.
- Lokshantov, Vatshelle and Villanger [5] found a polynomial-time algorithm to find a stable set of maximum weight in a P_5-free graph; and more recently Grzesik, Klimošová, Pilipczuk and Pilipczuk [4] claim to have the same for P_6-free graphs.