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List star chromatic index of sparse graphs

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ABSTRACT

A *star edge-coloring* of a graph G is a proper edge coloring such that every 2-colored connected subgraph of G is a path of length at most 3. For a graph G, let the *list star chromatic index* of G, $ch'_s(G)$, be the minimum k such that for any k-uniform list assignment L for the set of edges, G has a star edge-coloring from L. Dvořák et al. (2013) asked whether the list star chromatic index of every subcubic graph is at most 7. In Kerdjoudj et al. (2017) we proved that it is at most 8. In this paper we consider graphs with any maximum degree, we proved that if the maximum average degree of a graph G is less than $\frac{14}{5}$ (resp. 3), then $ch'_s(G) \leq 2\Delta(G) + 2$ (resp. $ch'_s(G) \leq 2\Delta(G) + 3$).

1. Introduction

All the graphs we consider are finite and simple. For a graph G, we denote by $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ its vertex set, edge set, minimum degree and maximum degree, respectively.

A *proper* vertex (respectively, edge) coloring of G is an assignment of colors to the vertices (respectively, edges) of G such that no two adjacent vertices (respectively, edges) receive the same color. A *star coloring* of G is a proper vertex coloring of G such that the union of any two color classes induces a *star forest* in G, i.e. every component of this union is a star. This notion was first mentioned by Grünbaum [7] in 1973 (see also [6]). The star coloring even in the class of line graphs seems to be difficult. A convenient language for discussions of star coloring of line graphs is the language of star edge-coloring of all graphs.

A *star edge-coloring* of a graph G is a proper edge-coloring such that every 2-colored connected subgraph of G is a path of length at most 3. In other words, we forbid bi-colored 4-cycles and 4-paths in G (by a k-path we mean a path with k edges). This notion is intermediate between *acyclic edge coloring*, when every 2-colored subgraph must be only acyclic, and *strong edge coloring*, when every 2-colored connected subgraph has at most two edges. The *star chromatic index* of G, denoted by $\chi'_{st}(G)$, is the minimum number of colors needed for a star edge-coloring of G. If we denote by $\chi'_s(G)$ the minimum number of colors needed for a strong edge-coloring of G. It is clear that $\chi'_{st}(G) \leq \chi'_s(G)$.

The star edge-coloring was first studied by Liu and Deng [11] in 2008. They proved the following upper bound.

Theorem 1.1 (*Liu and Deng '08 [11]*). *For every G with maximum degree $\Delta \geq 7$, $\chi'_{st}(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$.*

In [3] and later [1] it is proved:

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Theorem 1.2 (*Deng et al.'11 [3] and Bezegova et al.'16 [1]*). *The star chromatic index of any tree with maximum degree Δ is at most $\Delta + \lceil \frac{\Delta-1}{2} \rceil$.*

In [4], Dvořák, Mohar and Šámal considered the star chromatic index of the complete graph K_n with n vertices. They gave the following bounds:

$$2n(1+o(1)) \leq \chi'_{st}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log(n)}}}{\log n^{\frac{1}{4}}}.$$

In addition, they obtained the following near-linear upper bound given in terms of the maximum degree Δ.

Theorem 1.3 (*Dvořák, Mohar and Šámal '13 [4]*). *For every graph G of maximum degree Δ we have*

$$\chi'_{st}(G) \leq \chi'_{st}(K_{\Delta+1})O(\frac{\log \Delta}{\log \log \Delta})^2,$$

and therefore $\chi'_{st}(G) \leq \Delta.2^{O(1)\sqrt{\log \Delta}}$.

Note that no other upper bound is known for general graphs. They also studied the star chromatic index of *subcubic* graphs, that is, graphs with maximum degree at most 3. They proved that $\chi'_{st}(G) \leq 7$ for every subcubic graph G, and conjectured that $\chi'_{st}(G) \leq 6$ for every such G.

In [10] it is proved that it is NP-complete to determine whether $\chi'_{st}(G) \leq 3$ for an arbitrary graph.

A natural generalization of star edge-coloring is the list star edge-coloring. An *edge list* L for a graph G is a mapping that assigns a finite set of colors to each edge of G. Given an edge list L for a graph G, we say that G is *L-star edge-colorable* if it has a star edge-coloring c such that $c(e) \in L(e)$ for every edge of G. The *list-star chromatic index*, $ch'_s(G)$, of a graph G is the minimum k such that for every edge list L for G with $|L(e)| = k$ for every $e \in E(G)$, G is L-star edge-colorable. Dvořák, Mohar and Šámal [4, Question 3] asked whether $ch'_s(G) \leq 7$ for every subcubic G.

In [8] it is proved that:

Theorem 1.4 (*Kerdjoudj, Kostochka and Raspaud '17 [8]*). *For every subcubic graph G, $ch'_s(G) \leq 8$.*

We also gave sufficient conditions for the list-star chromatic index of a subcubic graph to be at most 5 and 6 in terms of the maximum average degree $\text{mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|},\ H \subseteq G\right\}$.

Theorem 1.5. *Let G be a subcubic graph.*

1. *If $mad(G) < \frac{7}{3}$ then $ch'_s(G) \leq 5$ (Kerdjoudj, Kostochka and Raspaud '17 [8]).*
2. *If $mad(G) < \frac{5}{2}$ then $ch'_s(G) \leq 6$ (Kerdjoudj, Kostochka and Raspaud '17 [8]).*
3. *If $mad(G) < \frac{30}{11}$ then $ch'_s(G) \leq 7$ (Kerdjoudj and Raspaud '18 [9]).*

In this paper we give upper bounds for the list-star chromatic index of sparse graphs in terms of the maximum degree. For the star chromatic index the bounds known are the ones given by the strong chromatic index since: $\chi'_{st}(G) \leq \chi'_s(G)$. In [2] it is proved:

Theorem 1.6 (*Choi, Kim, Kostochka and Raspaud '18 [2]*).

- *Let $\Delta \geq 9$ be an integer. Every graph G with $mad(G) < \frac{8}{3}$ and maximum degree at most Δ satisfies $\chi'_s(G) \leq 3\Delta - 3$.*
- *Let $\Delta \geq 7$ be an integer. Every graph G with $mad(G) < 3$ and maximum degree at most Δ satisfies $\chi'_s(G) \leq 3\Delta$.*

In a companion paper [9], we considered graphs with any maximum degree, we gave bounds of the list-star chromatic index of sparse graphs (i.e. graphs with a small maximum average degree):

Theorem 1.7 (*Kerdjoudj and Raspaud '18 [9]*). *Let G be a graph with maximum degree Δ.*

1. *If $mad(G) < \frac{7}{3}$ then $ch'_s(G) \leq 2\Delta - 1$.*
2. *If $mad(G) < \frac{5}{2}$ then $ch'_s(G) \leq 2\Delta$.*
3. *If $mad(G) < \frac{8}{3}$ then $ch'_s(G) \leq 2\Delta + 1$.*

The girth of a graph G is the size of a smallest cycle in G. As every planar graph with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$, Theorem 1.7 yields the following.

Corollary 1.1 (*Kerdjoudj and Raspaud '18 [9]*). *Let G be a planar graph with girth g.*

1. *If $g \geq 14$ then $ch'_s(G) \leq 2\Delta - 1$.*
2. *If $g \geq 10$ then $ch'_s(G) \leq 2\Delta$.*
3. *If $g \geq 8$ then $ch'_s(G) \leq 2\Delta + 1$.*