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Note

An improved bound for disjoint directed cycles

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ABSTRACT

We show that every directed graph with minimum out-degree at least $18k$ contains at least k vertex disjoint cycles. This is an improvement over the result of Alon who showed this result for digraphs of minimum out-degree at least $64k$. The main benefit of the argument is that getting better results for small values of k allows for further improvements to the constant.

1. Introduction

In this paper all digraphs are considered to contain no parallel edges, but loops and bidirectional edges (pairs of edges which join two vertices in opposite directions) are allowed, but considered as cycles of length 1 and 2 respectively. Throughout this paper by a cycle we mean a directed cycle.

We define $f(k)$ to be the minimal integer d for which any digraph G, with all vertices having at least d outgoing edges, contains k pairwise vertex disjoint cycles as subgraphs, if such a d does not exist we set $f(k) = \infty$.

The study of $f(k)$ was initiated by Bermond and Thomassen [4] who observed that as the complete directed graph[1] on $2k-1$ vertices has out-degree $2k-2$ and contains at most $k-1$ disjoint cycles, it follows that $f(k) \geq 2k-1$. They also conjecture that this is in fact tight, so that $f(k) = 2k-1$.

There has been plenty of work around this conjecture. Thomassen [7] showed that $f(2) = 3$ and that $f(k)$ is always finite. In particular, he showed $f(k) \leq (k+1)!$. Almost 15 years later Alon [1] improved significantly on this bound and showed that $f(k) \leq 64k$. More recently Pór and Sereni [6] showed that $f(3) = 5$. The conjecture has received attention even in the case when the digraph in question is restricted to be a tournament, with Bessy, Lichiardopol and Sereni [5] showing that it holds for tournaments with bounded minimum in-degree and later completely resolved, in case of tournaments, by Bang-Jensen, Bessy and Thomassé [3]

We improve on this bound to show $f(k) \leq 18k$. Our proof follows closely that of Alon [1], our main improvement is based on exploiting the fact that $f(3) = 5$, due to Pór and Sereni [6], which allows for significantly better bounds in a part of the argument.

1.1. Definitions and notation

For a vertex v of an undirected graph G we denote by $d_G(v)$ its degree and by $N_G(v)$ the set of its neighbours.

Given a digraph $G(V, E)$, for $x, y \in V$ we denote by xy the edge from x to y. If edge xy exists we say x is a *parent* of y and y is a *child* of x. If both edges xy and yx exist we say there is a *bidirectional* edge joining x and y.

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[1] A complete digraph has no loops, and between any two vertices, contains a bidirectional edge.

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We define the *out-degree* of a vertex v of a digraph G as the number of children of v in G and denote it as $d_G^+(v)$. We denote by $N_G^+(v)$ the set of children of v in G. We define the *in-degree* of a vertex v as the number of parents of v in G and denote it by $d_G^-(v)$ and we denote the set of these parents as $N_G^-(v)$. The index digraph G is omitted whenever there is no chance of confusion in regard to which graph is being referred to.

We say a digraph G is *k-out* if every vertex in G has out-degree at least k. A digraph is said to be *exactly k-out* if each out-degree in the graph equals k. *k-in* and *exactly k-in* graphs are defined analogously.

We denote by $V(G)$ the set of vertices of a (directed) graph G, for any $X \subseteq V(G)$ we denote by $G[X]$ the subgraph of G induced by X.

2. Preliminary results

We start by proving $f(2) = 3$, first proved by Thomassen [7]. We present a similar proof of this result in order to illustrate some of the ideas, which we will reuse later in the proof of our main result.

Lemma 2.1. $f(2) = 3$

Proof. A complete digraph on 3 vertices does not contain 2 disjoint cycles, implying $f(2) \geq 3$.

We now prove that any 3-out digraph contains 2 disjoint cycles. We proceed by induction on n, the number of vertices. For the base case we consider $n = 4$ where the only possible digraph is the complete digraph which contains 2 disjoint cycles of length 2. We now assume that any 3-out digraph, with $n - 1 \geq 4$ vertices, contains 2 disjoint cycles.

Assuming there is a digraph on n vertices failing our assumption, we can remove some of its edges to make it exactly 3-out, such new digraph still does not contain 2 disjoint cycles. We call this digraph G.

G has no bidirectional edges. If uw is a bidirectional edge then $G - \{u, w\}$ is still 1-out so it contains a cycle, which paired with the 2-cycle made by the bidirectional edge uw gives the desired disjoint cycles.

The main idea allowing us to prove the result is using edge contractions. If there is an edge uv such that u, v have no common parent, we can modify G to a new digraph G' by removing u and v and adding a vertex w having $N_{G'}^+(w) \equiv N_G^+(v)$ and $N_{G'}^-(w) \equiv N_G^-(u) \cup N_G^-(v)$. G' has $n - 1$ nodes and is still 3-out, as u, v had no common parent, so by the inductive assumption G' contains 2 disjoint cycles. If w is not in any of the cycles they were contained in G to start with and we are done. The other option is that w is contained in one of the cycles, this implies the other cycle is in G. We distinguish the cases when the cycle in-edge of w comes from u's in-edge or from v's in-edge. Replacing w in the first case by uv and in the second by v we get the other cycle in G and we are done.

The only remaining option is when every edge of G has a *witness* (defined as common parent to both its end-vertices).

Let v be a vertex having the smallest in-degree. As there are $3n$ edges in total we have $d^-(v) \leq 3$.

- Case 1: $d^-(v) = 0$ Then $G - \{v\}$ is 3-out and has $n - 1$ vertices so inductively we can find disjoint cycles which are also contained in G.
- Case 2: $d^-(v) = 1$ The edge ending in v has no witnesses.
- Case 3: $d^-(v) = 2$ Let $N^-(v) \equiv \{u, w\}$. u has to be the witness to wv and w to uv implying uw is a bidirectional edge which is a contradiction.
- Case 4: $d^-(v) = 3$ By our choice of v, for all $x \in V(G)$ we have $d^-(x) \geq d^-(v) = 3$ and $3n = \sum_{x \in V(G)} d^-(x) \geq nd^-(v) = 3n$ so we need to have equality in $d^-(x) \geq d^-(v)$ for all x implying all vertices have in-degree 3. Given a vertex x and its parents u, v, w we show that u, v, w make a 3-cycle. Indeed, we notice that each of the witnesses of ux, vx, wx must be among u, v, w implying that the u, v, w induced subgraph is 1-in. This combined with the fact bidirectional edges do not exist, we conclude that u, v, w make a 3-cycle as claimed. If we reverse the edges of G we notice that all the above arguments still apply, as G is both exactly 3-out and exactly 3-in, so children of x also form a 3-cycle. As there are no bidirectional edges, the children and parents of x are disjoint and give 2 disjoint 3-cycles. □

3. The main result

As noted before considering the complete digraph on $2k - 1$ vertices we have $f(k) \geq 2k - 1$.

We start by defining a class of, in some sense, minimal counterexamples to the Bermond–Thomassen conjecture. The main reason for this is to fix a minor flaw in the way the argument of Alon in [1] is written, where he looks at a minimal counterexample to $f(k) \leq 64k$ and shows Proposition 3.2 for it, but then also uses this proposition for graphs which are minimal counterexamples to different inequalities and as a consequence omits a rather easy case, specifically the second case of Corollary 3.8.

For positive integers r, k we say a digraph G is (k, r)-*critical* if the following properties hold:

(1) G is r-out,
(2) G does not contain $k + 1$ disjoint cycles,
(3) Subject to properties (1) and (2) G has the smallest number of vertices and
(4) Subject to properties (1), (2) and (3) G has the smallest possible number of edges.
(5) Any $(r - 2)$-out digraph contains k disjoint cycles.