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Combinatorial interpretations of mock theta functions by attaching weights

S. Sharma, M. Rana *

School of Mathematics, Thapar Institute of Engineering and Technology, Patiala 147004, India

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ABSTRACT

This paper provides the combinatorial interpretations of twelve mock theta functions in terms of $(n + t)$-color partitions with attached weights. We further find combinatorial interpretations of the generalized versions of all these mock theta functions and present these into six families.

1. Introduction

Mock theta functions came into existence with the last letter of Ramanujan to Hardy in 1920. He gave seventeen mock theta functions separated into three classes as per order. This number kept on increasing with the discovery of mock theta functions by other mathematicians (see Watson [24,25], Garvan [15,14], Gordon–McIntosh [16], Lovejoy [18]).

A number of mock theta functions have been studied analytically as well as combinatorially by several mathematicians (see Fine [13], Choi [11], Agarwal [3], Agarwal and Rana [20], Agarwal and Narang [5]). However, the study of mock theta functions in terms of $(n + t)$-color partitions started with the interpretations of the following four mock theta functions by Agarwal [2]:

$$\psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q; q^2)_m}, \tag{1}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q; q^2)_m}, \tag{2}$$

$$\phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q; q^2)_m, \tag{3}$$

and

$$\phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q; q^2)_m, \tag{4}$$

* Corresponding author.
E-mail address: mrana@thapar.edu (M. Rana).

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where, $(n+t)$-color partitions are defined as follows:

Definition 1 (*[4]*). An $(n+t)$-color partition (also called a partition with "$n+t$ copies of n"), $t \geq 0$ is a partition in which a part of size n, $n \geq 0$ can come in $n+t$ different colors denoted by $n_1, n_2, \ldots, n_{n+t}$. Note that zeros are permitted, if and only if $t > 0$, but zeros cannot repeat. The weighted difference of two parts $m_i, n_j,\ m \geq n$ is defined by $m - n - i - j$ and denoted by $((m_i - n_j))$.

In continuation of work done in [2], Agarwal and Rana [20] interpreted the following fifth order mock theta function in terms of $(n+2)$-color partitions:

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}}. \tag{5}$$

Sareen and Rana [22] further studied the following two mock theta functions of order 10 in terms of $(n+t)$-color partitions:

$$\phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q; q^2)_{n+1}}, \tag{6}$$

$$\psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}}}{(q; q^2)_{n+1}}. \tag{7}$$

To handle the following mock theta functions of order 8 with $(n+t)$-color partitions, Agarwal and Sood [7] split the color of partitions into two parts and named these partitions as split $(n+t)$-color partitions:

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q; q^2)_n}, \tag{8}$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{(n+1)^2}}{(q; q^2)_{n+1}}. \tag{9}$$

One can easily see that, in the expansion of all the mock theta functions from (1)–(9), the factors generate only positive coefficients in powers of q. However, Brietzke et al. [10] have interpreted a number of mock theta functions in terms of two-line arrays [21], which are having some negative coefficients in the expansion. To arrive at the interpretations of such mock theta functions, they attached certain weights to each two-line array generated by the unsigned version of the mock theta function being interpreted. Inspiring from their work we provide the interpretations of a number of mock theta functions by attaching weights to the $(n+t)$-color partitions generated by the unsigned version of these mock theta functions. We interpret the following two mock theta functions of order 3 from Ramanujan's last letter to Hardy [8]:

$$\chi(q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n^2}}{(-q^3; q^3)_n}, \tag{10}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n}. \tag{11}$$

Another mock theta function, which becomes a part of our study, is the following function added by Watson [24] to the list of third order mock theta functions:

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}}. \tag{12}$$

From the list of Ramanujan's fifth order mock theta functions, we study the following two:

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}, \tag{13}$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q)_n}. \tag{14}$$

The only sixth order mock theta function being studied here is the following one which appears along with other sixth and tenth order mock theta functions in Ramanujan's lost notebook [19]:

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{(q; q)_n q^{n^2}}{(q^3; q^3)_n}. \tag{15}$$