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Not-all-equal 3-SAT and 2-colorings of 4-regular 4-uniform hypergraphs

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ABSTRACT

In this paper, we continue our study of 2-colorings in hypergraphs (see, Henning and Yeo, 2013). A hypergraph is 2-colorable if there is a 2-coloring of the vertices with no monochromatic hyperedge. It is known (see Thomassen, 1992) that every 4-uniform 4-regular hypergraph is 2-colorable. Our main result in this paper is a strengthening of this result. For this purpose, we define a vertex in a hypergraph H to be a free vertex in H if we can 2-color $V(H) \setminus \{v\}$ such that every hyperedge in H contains vertices of both colors (where v has no color). We prove that every 4-uniform 4-regular hypergraph has a free vertex. This proves a conjecture in Henning and Yeo (2015). Our proofs use a new result on not-all-equal 3-SAT which is also proved in this paper and is of interest in its own right.

1. Introduction

In this paper, we continue the study of 2-colorings in hypergraphs. We adopt the notation and terminology from [5,6]. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of arbitrary subsets of V, called *hyperedges* or simply *edges*. A k-edge in H is an edge of size k in H. The hypergraph H is *k-uniform* if every edge of H is a k-edge. The *degree* of a vertex v in H, denoted $d_H(v)$ or simply by $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The hypergraph H is *k-regular* if every vertex has degree k in H. For $k \geq 2$, let $\mathcal{H}_k$ denote the class of all k-uniform k-regular hypergraphs. The class $\mathcal{H}_k$ has been studied, both in the context of solving problems on total domination as well as in its own right, see for example [1,3–7,13].

A hypergraph H is 2-*colorable* if there is a 2-coloring of the vertices with no monochromatic hyperedge. Equivalently, H is 2-colorable if it is *bipartite*; that is, its vertex set can be partitioned into two sets such that every hyperedge intersects both partite sets. Alon and Bregman [1] established the following result.

Theorem 1 (*Alon, Bregman [1]*). *Every hypergraph in $\mathcal{H}_k$ is 2-colorable, provided $k \geq 8$.*

Thomassen [15] showed that the Alon–Bregman result in Theorem 1 holds for all $k \geq 4$.

Theorem 2 (*Thomassen [15]*). *Every hypergraph in $\mathcal{H}_k$ is 2-colorable, provided $k \geq 4$.*

As remarked by Alon and Bregman [1] the result is not true when $k = 3$, as may be seen by considering the Fano plane. Sufficient conditions for the existence of a 2-coloring in k-uniform hypergraphs are given, for example, by Radhakrishnan

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and Srinivasan [11] and Vishwanathan [16]. For related results, see the papers by Alon and Tarsi [2], Seymour [12] and Thomassen [14].

A set X of vertices in a hypergraph H is a *free set* in H if we can 2-color $V(H)\setminus X$ such that every edge in H contains vertices of both colors (where the vertices in X are not colored). A vertex is a *free vertex* in H if we can 2-color $V(H) \setminus \{v\}$ such that every hyperedge in H contains vertices of both colors (where v has no color). In [6] it is conjectured that every hypergraph $H \in \mathcal{H}_k$, with $k \geq 4$, has a free set of size $k-3$. Further, if the conjecture is true, then the bound $k-3$ cannot be improved for any $k \geq 4$, due to the complete k-uniform hypergraph of order $k+1$, as such a hypergraph needs two vertices of each color to ensure every edge has vertices of both colors. The conjecture is proved to hold for $k \in \{5, 6, 7, 8\}$. The case when $k = 4$ turned out to be more difficult than the cases when $k \in \{5, 6, 7, 8\}$ and was conjectured separately in [6].

Conjecture 1 (*[6]*). *Every* 4*-regular* 4*-uniform hypergraph contains a free vertex.*

2. Terminology and definitions

For an edge e in a hypergraph H, we denote by $H - e$ the hypergraph obtained from H by deleting the edge e. Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. Further, x and y are *connected* if there is a sequence $x = v_0, v_1, v_2 \dots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \dots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A *component* of a hypergraph H is a maximal connected subhypergraph of H. In particular, we note that a component of H is by definition connected.

A subset T of vertices in a hypergraph H is a *transversal* in H if T has a nonempty intersection with every edge of H. In the language of transversals, a vertex v is a free vertex in a hypergraph H if H contains two vertex disjoint transversals, neither of which contain the vertex v. Transversals in 4-uniform hypergraphs are well studied (see, for example, [6,9,13]).

In order to prove Conjecture 1, we use a connection between an instance of not-all-equal 3-SAT (NAE-3-SAT) and a 3-uniform hypergraph. In order to state this connection we require some further terminology.

Definition 1. An instance, I, of 3-SAT contains a set of variables, $V(I)$, and a set of clauses, $C(I)$. Each clause contains exactly three literals, which are either a variable, $v \in V(I)$, or the negation of a variable, $\overline{v}$, where $v \in V(I)$. A clause, $c \in C(I)$, is satisfied if one of the literals in it is true. That is, the clause c is satisfied if there exists $v \in V(I)$ that belongs to c and $v = True$ or $\overline{v}$ belongs to c and $v = False$. The instance I is satisfied if there is a truth assignment to the variables such that all clauses are satisfied.

Definition 2. An instance of NAE-3-SAT is equivalent to 3-SAT, except that we require all clauses to contain a false literal as well as a true one. A clause that contains both a true and false literal we call *nae-satisfiable*. If there is a truth assignment to the variables such that every clause in the instance I is (simultaneously) nae-satisfiable, we say that I is *nae-satisfiable*. An instance of NAE-3-SAT is non-trivial if it contains at least one variable.

We remark that NAE-3-SAT is a known decision problem in theoretical computer science, and its NP-completeness can be proven by a reduction from 3-SAT as shown by Moore and Mertens [10]. We furthermore need the following definitions.

Definition 3. Given an instance I of NAE-3-SAT, we define the **associated graph** G_I to be the graph whose vertices correspond to the variables in the set $V(I)$ and where an edge joins two vertices in G_I if the associated variables (either in negated or unnegated form) appear in the same clause in I.

Let I be an instance of NAE-3-SAT. We call the instance I **connected** if one cannot partition the variables $V(I)$ into non-empty sets V_1 and V_2 such that no clause contains variables from V_1 and V_2. In other words, the graph G_I associated with I is connected.

A **component** of a NAE-3-SAT instance I is a maximal connected sub-instance of I. That is, the components of I correspond precisely to the components of the graph G_I associated with I.

A variable, $v \in V(I)$, is **free** if I is nae-satisfiable even if we do not assign any truth value to v. That is, every clause in I contains a true and a false literal, even without considering literals involving v.

The **degree** of a variable $v \in V(I)$, is the number of clauses containing v or $\overline{v}$, and is denoted by $\deg_I(v)$. If the instance I is clear from the context, we simply write $\deg(v)$ rather than $\deg_I(v)$.

We are now in a position to define a connection between an instance of NAE-3-SAT and a 3-uniform hypergraph as follows.

Definition 4. If H is a 3-uniform hypergraph, we create a NAE-3-SAT instance I_H as follows. Let $V(I_H) = V(H)$ and for each edge $e \in H$ add a clause to I_H with the same vertices/variables in non-negated form. We call I_H the NAE-3-SAT instance corresponding to H. Note that the instance I_H is nae-satisfiable if and only if H is bipartite. In fact the partite sets in the bipartition correspond to the truth values true and false.

Throughout this paper, we use the standard notation $[k] = \{1, 2, \dots, k\}$.