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Balanced diagonals in frequency squares

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ABSTRACT

We say that a diagonal in an array is λ-*balanced* if each entry occurs λ times. Let L be a frequency square of type $F(n; \lambda)$; that is, an $n \times n$ array in which each entry from $\{1, 2, \ldots, m = n/\lambda\}$ occurs λ times per row and λ times per column. We show that if $m \leqslant 3$, L contains a λ-balanced diagonal, with only one exception up to equivalence when $m = 2$. We give partial results for $m \geqslant 4$ and suggest a generalization of Ryser's conjecture, that every Latin square of odd order has a transversal. Our method relies on first identifying a small substructure with the frequency square that facilitates the task of locating a balanced diagonal in the entire array.

1. Introduction

In what follows, rows and columns of an $n \times n$ array L are each indexed by $N(n) = \{1, 2, \ldots, n\}$, with $L_{i,j}$ denoting the *entry* in cell (i, j). We sometimes consider an array L to be a set of ordered triples $L = \{(i, j; L_{i,j})\}$ so that the notion of a subset of an array is precise. A subarray of L is any array induced by subsets of the rows and columns of L; thus the rows and columns in a subarray need not be adjacent.

A *frequency square* or F-*square* L of *type* $F(n; \lambda_1, \lambda_2, \ldots, \lambda_m)$ is an $n \times n$ array such that for each $i \in N(m)$, i occurs λ_i times in each row and λ_i times in each column; necessarily $\sum_{i=1}^{m} \lambda_i = n$. In the case where $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda$ we say that L is of type $F(n; \lambda)$; unless otherwise stated $m = n/\lambda$. Clearly a frequency square of type $F(n; 1)$ is a Latin square of order n.

We define a *diagonal* in any square array to be a subset that uses each row and each column exactly once. We say that a diagonal is λ-balanced if each entry occurs λ times, for some λ. Thus, a 1-balanced diagonal in a Latin square is precisely a transversal.

In this paper we restrict ourselves to frequency squares of type $F(n; \lambda)$; in this context we refer to a λ-balanced diagonal as simply being *balanced*; here each element of $N(m)$ appears exactly λ times. For our purposes, two frequency squares of type $F(n; \lambda)$ are equivalent if and only if one can be obtained from the other by rearranging rows or columns, relabelling symbols or taking the transpose.

Trivially, any diagonal of a frequency square of type $F(\lambda; \lambda)$ is balanced. In Section 2 we show, with one exception up to equivalence, that each frequency square of type $F(2\lambda; \lambda)$ has a balanced diagonal. In Section 3 we show that *every* frequency square of type $F(3\lambda; \lambda)$ has a balanced diagonal. We then make some observations and conjectures about the existence of balanced diagonals in $F(m\lambda; \lambda)$ for $m > 3$ in Section 4.

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The existence of *transversals* in arrays (diagonals in which each entry appears at most once) or, equivalently, *rainbow matchings* in coloured bipartite graphs, has been well-studied [1,14]. However there appears to be scarce results on the existence of diagonals in which each entry has a fixed number of multiple occurrences. Nevertheless, the existence of transversals (and other regular structures called *plexes*) in Latin squares imply the existence of balanced diagonals in certain frequency squares, as shown in Section 4. Conversely, the results in this paper suggest a certain generalization of Ryser's conjecture, that each Latin square of odd order has a transversal; see Conjecture 9.

Two frequency squares are said to be *orthogonal* if, when superimposed, each ordered pair occurs a constant number of times. Research into frequency squares focuses mainly on constructing sets of mutually orthogonal frequency squares (MOFS). The maximum possible size of a set of MOFS of type $F(n;\lambda)$ is $(n-1)^2/(m-1)$ [7]; such a set is called *complete*. There are various constructions for complete sets of MOFS of type $F(n;\lambda)$ in the literature [8–10]; it was shown by Mavron that all such sets of MOFS can be derived from affine designs [11].

The relationship between the existence of a balanced diagonal in a frequency square F and whether that frequency square is orthogonal to another frequency square appears to the authors not to be trivial; we leave this as an open question for exploration. Certainly if L is a frequency square of type $F(n;\lambda)$ and L is orthogonal to a Latin square of order n, then L must partition into balanced diagonals (the cells of a fixed entry in L form a balanced diagonal in $F(n;\lambda)$). The frequency square A_6 of type $F(6;3)$ below is shown in the next section to have no balanced diagonals, yet is orthogonal to the following frequency square B below of type $F(6;2)$.

1	1	1	2	2	2
1	1	1	2	2	2
1	1	1	2	2	2
2	2	2	1	1	1
2	2	2	1	1	1
2	2	2	1	1	1

A_6

1	2	3	1	2	3
2	3	1	2	3	1
3	1	2	3	1	2
1	2	3	1	2	3
2	3	1	2	3	1
3	1	2	3	1	2

B

Instead of starting with any diagonal and trying to permute rows and columns to make it balanced, we obtain our main results in Sections 2 and 3 by first identifying a subarray that allows us to construct a diagonal within the rest of the square that is *close* to being balanced. The properties of the subarray then allow us to find a balanced diagonal in the entire square. This approach makes the proof of Theorem 2 in particular delightfully terse (compared to an originally drafted much longer proof) and the proof of Theorem 3 manageable. This idea may be of use towards the solution of related combinatorial problems.

2. Balanced diagonals in frequency squares with 2 symbols

Let $A_{2\lambda}$ be the frequency square of type $F(2\lambda;\lambda)$ with only 1's in the top-left and bottom-right quadrants, formally defined as follows:

$$A_{2\lambda} = \{(i,j;1),(i+\lambda,j+\lambda;1),(i,j+\lambda;2),(i+\lambda,j;2) \mid i,j \in N(\lambda)\}.$$

The frequency square A_6 was given in the Introduction.

Lemma 1. *The frequency array $A_{2\lambda}$ possesses a balanced diagonal if and only if λ is even.*

Proof. It is easy to find a balanced diagonal if λ is even. If λ is odd, suppose that $A_{2\lambda}$ possesses a balanced diagonal M with x elements in cells (i,j) where $i,j\in N(\lambda)$. Then M has $\lambda-x$ elements in cells (i,j) where $i,j-\lambda\in N(\lambda)$ and in turn, x elements in cells (i,j) where $i-\lambda, j-\lambda \in N(\lambda)$. Thus, $2x$ elements of M contain entry 1, contradicting the fact that λ is odd. □

As an aside we note that if λ is odd, $A_{2\lambda}$ is not orthogonal to any frequency square of type $F(2\lambda;\lambda)$. The proof is very similar to the previous proof.

Theorem 2. *Let L be a frequency square of type $F(2\lambda;\lambda)$. Then L has a balanced diagonal, unless L is equivalent to $A_{2\lambda}$ where λ is odd.*

Proof. Let L be a frequency square of type $F(2\lambda;\lambda)$. Observe that if L does not possess the following subarray, it must be equivalent to $A_{2\lambda}$ and the previous lemma applies.

1	1
1	2

Otherwise, we assume without loss of generality that $L_{1,1}=L_{1,2}=L_{2,1}=1$ and $L_{2,2}=2$. Let M be the main diagonal and let x be the number of 1's in M. Rearrange the rows and columns (except for the first two rows and columns) so that $|x-\lambda|$ is minimized. If $x-\lambda=0$ the main diagonal is balanced and we are done. If $x-\lambda=-1$, we can swap rows 1 and 2 and the main diagonal becomes a balanced diagonal.