Discrete Mathematics 341 (2018) 1932–1944

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Diameter bounds and recursive properties of Full-Flag Johnson graphs

Irving Dai

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

ARTICLE INFO

Article history:
Received 6 July 2017
Received in revised form 9 January 2018
Accepted 15 March 2018

Keywords:
Johnson graph
Permutahedron
Irreducible permutation
Cayley graph
Diameter
Spectrum

ABSTRACT

The Johnson graphs $J(n, k)$ are a well-known family of combinatorial graphs whose applications and generalizations have been studied extensively in the literature. In this paper, we present a new variant of the family of Johnson graphs, the Full-Flag Johnson graphs, and discuss their combinatorial properties. We show that the Full-Flag Johnson graphs are Cayley graphs on S_n generated by certain well-known classes of permutations and that they are in fact generalizations of permutahedra. We prove a tight $\Theta(n^2/k^2)$ bound for the diameter of the Full-Flag Johnson graph $FJ(n, k)$ and establish recursive relations between $FJ(n, k)$ and the lower-order Full-Flag Johnson graphs $FJ(n-1, k)$ and $FJ(n-1, k-1)$. We apply this recursive structure to partially compute the spectrum of permutahedra.

1. Introduction

For positive integers n and k with $k < n$, the Johnson graph $J(n, k)$ is an undirected graph whose vertex set is given by the collection of all k-element subsets of $\{1, 2, \ldots, n\}$. Two vertices are adjacent if and only if their intersection has size $k - 1$. For example, two vertices in $J(4, 3)$ are $u = \{1, 2, 3\}$ and $v = \{1, 3, 4\}$; u and v are adjacent since $|u \cap v| = 2$. The Johnson graphs are known to be Hamiltonian [2] and their spectra are given by the Eberlein polynomials [12] (see also [19]). Different generalizations of Johnson graphs and various related families have been studied by several authors; see for example [1,3,9,7,21].

The permutahedra are another well-known family of combinatorial graphs. For a positive integer n, the permutahedron of order n has vertex set consisting of all permutations of $(1, 2, \ldots, n)$. Two vertices are adjacent if and only if they are of the form $(u_1, u_2, \ldots, u_i, u_{i+1}, \ldots, u_n)$ and $(u_1, u_2, \ldots, u_{i+1}, u_i, \ldots, u_n)$, respectively. That is, u and v are adjacent if they are related by a permutation which transposes two consecutive elements. (We call such transpositions "neighboring transpositions".) Permutahedra appear frequently in geometric combinatorics and are Hamiltonian by the Steinhaus–Johnson–Trotter algorithm [16]. Similar combinatorial families of graphs (in particular, associahedra) and their generalizations appear widely in algebra and discrete mathematics (see e.g. [8,23]).

In this paper, we present and discuss some combinatorial properties of a new variant of the set of Johnson graphs, the Full-Flag Johnson graphs. Roughly speaking, the Full-Flag Johnson graphs are constructed by imposing (an index-shifted version of) the Johnson graph adjacency condition on the collection of all full-flags of $\{1, 2, \ldots, n\}$. We show that Full-Flag Johnson graphs are Cayley graphs on the symmetric group S_n generated by certain classes of permutations, and that they are in fact generalizations of permutahedra.

Our first significant result will be to derive a tight $\Theta(n^2/k^2)$ bound for the diameter of the Full-Flag Johnson graph $FJ(n, k)$. Several authors have studied bounds for the diameters of generalized Johnson graphs and associahedra; see for example

E-mail address: idai@math.princeton.edu.

https://doi.org/10.1016/j.disc.2018.03.014
0012-365X/

Pournin on the diameter of associahedra [24], Bautista-Santiago et al. on the diameter of generalized Johnson graphs [6] and Manneville and Pilaud on the graph-theoretic properties of graph associahedra [20]. In our case, we take a rather different approach to bounding the diameter of $FJ(n, k)$ by translating the question into the *minimum-length generator problem*: given a permutation $\sigma \in S_n$ and fixed subset S of S_n, how many applications of elements of S suffice to sort σ? As might be expected, in general this problem is difficult (NP-hard [13]). Bounds for specific instances of S have been studied widely in the literature; for example: sorting by reversals, cyclic transpositions [15], block transpositions [4], bounded-block transpositions [14], and so on. It turns out that for the class of permutations at hand, utilizing a parallel sorting algorithm of Baudet and Stevenson [5] allows us to easily obtain an upper bound on the number of generators needed and thus the diameter of $FJ(n, k)$.

We also prove recursive relations between the adjacency matrices of $FJ(n, k)$ and the lower-order Full-Flag Johnson graphs $FJ(n-1, k)$ and $FJ(n-1, k-1)$. We apply these in the case $k = 1$ to partially compute the spectrum of permutahedra.

The first part of this paper was presented at the International Workshop on Combinatorial Algorithms (Verona, Fall 2015); Sections 1 through 5 appeared in the conference proceedings [11]. The author would like to thank the conference organizers and referees for their helpful comments and remarks. Various introductory results and lemmas were proven jointly with Michael Greenberg, Noah Schoem, and Matt Tanzer at the Program in Mathematics for Young Scientists (Boston University, Summer 2010). The author would like to thank Paul Gunnells for formulating and proposing that project, out of which this paper eventually grew. The author would also like to thank Ho-Kwok Dai for various helpful conversations and ideas during the course of writing this paper. This work was carried out with the support of NSF grant number DGE 1148900.

2. Definitions and examples

Let n be a positive integer. Denote by $[n]$ and (n) the unordered and ordered sets $\{1, 2, \dots, n\}$ and $(1, 2, \dots, n)$, respectively. A *full-flag of subsets* of $[n]$ is a sequence of nested subsets $U = (U_1, U_2, \dots, U_n)$ of $[n]$ such that $|U_i| = i$ for all $i \in [n]$ and $U_i \subsetneq U_{i+1}$ for all $i \in [n-1]$. For example, one full-flag of subsets of $\{1, 2, 3, 4\}$ is $(\{3\}, \{3, 1\}, \{3, 1, 2\}, \{3, 1, 2, 4\})$. For a non-negative integer k such that $k < n$, the *Full-Flag Johnson graph* $FJ(n, k)$ has vertex set given by the collection of all possible full-flags of $[n]$. Two vertices $U = (U_1, U_2, \dots, U_n)$ and $V = (V_1, V_2, \dots, V_n)$ are adjacent in $FJ(n, k)$ if and only if $U_i \neq V_i$ for exactly k integers $i \in [n]$. If we view U and V as collections of subsets of $[n]$, then U and V are adjacent if and only if $|U \cap V| = n - k$.

The following equivalent definition of $FJ(n, k)$ simplifies the vertex set at the expense of complicating the adjacencies. Let $U = (U_1, U_2, \dots, U_n)$ be any vertex in $FJ(n, k)$. For each $1 < i \leq n$, the difference $U_i - U_{i-1}$ is a singleton element which we denote by u_i. Letting u_1 be the singleton element of U_1, we may identify U uniquely with the sequence $u = (u_1, u_2, \dots, u_n)$. It is clear that u must be a permutation of (n) and that $U_i = \{u_1, u_2, \dots, u_i\}$. Since every permutation of (n) corresponds to a full-flag of subsets of $[n]$ in this manner, we may view the vertex set of $FJ(n, k)$ as the collection of all permutations of (n). Two vertices $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ are adjacent if and only if there exist exactly k positive integers $i \in [n]$ such that $\{u_1, u_2, \dots, u_i\} \neq \{v_1, v_2, \dots, v_i\}$.

Example 1. Let $u = (1, 2, 3, 4, 5)$ and $v = (2, 1, 3, 5, 4)$ be two vertices in $FJ(5, 2)$. Then u and v are adjacent since $\{u_1, u_2, \dots, u_i\} \neq \{v_1, v_2, \dots, v_i\}$ for exactly two values of $i \in [5]$ ($i = 1$ and $i = 4$).

Given any permutation $u = (u_1, u_2, \dots, u_n)$ of (n), we use $u(i)$ to denote the set $\{u_1, u_2, \dots, u_i\}$, with the understanding that $u(0)$ is the empty set. Some algebraic rules for these sets are immediately evident. For instance, since the elements of the sequence u are distinct, for all x and y such that $0 \leq x < y \leq n$, we have $u(y) - u(x) = \{u_{x+1}, u_{x+2}, \dots, u_{y-1}, u_y\}$. Thus, if $u(x) = v(x)$ and $u(y) = v(y)$, then clearly $u(y) - u(x) = v(y) - v(x)$.

It is easily seen that the Full-Flag Johnson graph $FJ(n, 0)$ is the isolated graph, in which each vertex is adjacent only to itself. Less trivially, we have:

Lemma 1. *The graph $FJ(n, 1)$ is the permutahedron of order n, in which two vertices are adjacent if and only if they are related by a neighboring transposition.*

Proof. Let n be any positive integer. Two vertices u and v are adjacent in $FJ(n, 1)$ if and only if $u(i) \neq v(i)$ for exactly one $i \in [n]$. In particular, $u(x) = v(x)$ for each positive integer $x < i$, which implies $u_1 = v_1, u_2 = v_2, \dots, u_{i-1} = v_{i-1}$. Since $u(i-1) = v(i-1)$ but $u(i) \neq v(i)$, we have $u_i \neq v_i$. Hence the first $i-1$ elements of the sequence u are equal to the first $i-1$ elements of the sequence v, and the ith elements of u and v differ.

Now, it cannot be that $i = n$, since $u(n) = v(n) = \{1, 2, \dots, n\}$. Thus $i < n$, and $u(i+1) = v(i+1)$. Since $u(i-1) = v(i-1)$, we then have $\{u_i, u_{i+1}\} = \{v_i, v_{i+1}\}$. But $u_i \neq v_i$, so it must be that $v_i = u_{i+1}$ and $v_{i+1} = u_i$.

Finally, $u(x) = v(x)$ for all $x > i$, which implies $u_{i+2} = v_{i+2}, u_{i+3} = v_{i+3}, \dots, u_n = v_n$. Hence u and v are of the form

$$u = (u_1, u_2, \dots, u_i, u_{i+1}, \dots, u_n) \text{ and}$$
$$v = (u_1, u_2, \dots, u_{i+1}, u_i, \dots, u_n),$$

respectively; that is, they are related by a neighboring transposition. Conversely, it is evident that two vertices related by a neighboring transposition are adjacent in $FJ(n, 1)$. □