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Reconstruction of distance hereditary 2-connected graphs

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ABSTRACT

A graph is *reconstructible* if it is determined up to isomorphism from the collection of all its one-vertex deleted unlabelled subgraphs. It is shown that all distance hereditary 2-connected graphs G such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

1. Introduction

All graphs considered in this paper have finite orders and do not have loops or multiple edges. The terms not defined here are taken as in [7]. The *distance* between two vertices u and v in a connected graph G, denoted $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path joining them. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the maximum of its distances to other vertices. The *radius* and *diameter* of a connected graph G, denoted $rad(G)$ and $\text{diam}(G)$, respectively, are the minimum and maximum of the vertex eccentricities, respectively. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A graph G is *k-connected* if $\kappa(G) \geq k$. The *complement* $\overline{G}$ of a graph G is the graph having the same vertex set as G and uw is an edge of $\overline{G}$ if and only if it is not an edge of G.

A *vertex-deleted subgraph* (or *card*) $G - v$ of a graph G is the unlabelled subgraph obtained from G by deleting the vertex v and all edges incident with v. The collection of all cards of G is the *deck* of G. A graph H is a *reconstruction* of G if H has the same deck as G. A graph is *reconstructible* if it is isomorphic to all its reconstructions. A family $\mathscr{F}$ of graphs is *recognizable* if, for each $G \in \mathscr{F}$, every reconstruction of G is also in $\mathscr{F}$, and *weakly reconstructible* if, for each graph $G \in \mathscr{F}$, all reconstructions of G that are in $\mathscr{F}$ are isomorphic to G. A family $\mathscr{F}$ of graphs is *reconstructible* if, for any graph $G \in \mathscr{F}$, G is reconstructible (i.e. if $\mathscr{F}$ is both recognizable and weakly reconstructible). A parameter p defined on graphs is reconstructible if, for any graph G, it takes the same value on every reconstruction of G. The graph reconstruction conjecture (RC), posed by Kelly and Ulam in 1941 (see [4]), says that every graph G on n ($\geq$3) vertices is reconstructible. This conjecture has been proved notoriously difficult, and has motivated a large amount of work in graph theory. The manuscripts [4], [5], [8], [13] and [14] are surveys of work done on this problem.

Yang Yongzhi [21] settled Problem 3 listed in the survey [5] when he proved that every connected graph is reconstructible if and only if every 2-connected graph is reconstructible. Gupta et al. [10] have proved that the RC is true if and only if all connected graphs G such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible. Ramachandran and Monikandan [18] have combined their results and proved that the RC is true if and only if all 2-connected graphs G such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

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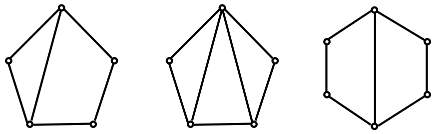

(a) (b) (c)

Fig. 1. Forbidden subgraphs.

A graph G is *distance-hereditary* if for all connected induced subgraphs F of G, $d_F(u, v) = d_G(u, v)$ for every pair of vertices $u, v \in V(F)$. Distance-hereditary graphs were introduced by Howorka [11], who was also the first to characterize these graphs. He proved that a connected graph G is distance-hereditary if and only if every cycle in G of at least five vertices has a pair of diagonals that cross each other. Thus, block graphs (connected graphs in which every block (maximal connected subgraph containing no cut vertices) is complete) and cographs (graphs containing no induced path of four vertices) are distance-hereditary. An important subclass of the distance-hereditary graphs is the Ptolemaic graphs. A graph G is *Ptolemaic* if for any four vertices x, y, z, w of G, the Ptolemy inequality $d(x, y)\, d(z, w) \leq d(x, z)\, d(y, w) + d(x, w)\, d(y, z)$ holds. It was shown in [12] that G is Ptolemaic if and only if G is distance-hereditary and chordal (graphs in which every cycle of at least four vertices has a chord). In their paper [2], Bandelt and Mulder gave a positive characterization of distance-hereditary graphs, which was the first of its kind. They proved that a nontrivial connected graph G is distance-hereditary if and only if G can be obtained from K_2 by a sequence of following operations:

(i) Adding a new vertex v' and joining it only to one vertex v.
(ii) Adding a new vertex v' and joining it to some vertex v and all its neighbours.
(iii) Adding a new vertex v' and joining it to the neighbours of some vertex v but not to v.

Yeh and Chang [20] investigated the relations between eccentricity, radius, and diameter of these graphs. They proved that the centre of a distance-hereditary graph is either a connected graph with diameter at most three or a cograph. During the last four decades since the introduction by Howorka, several characterizations and properties of distance-hereditary graphs were obtained, see Refs. [1,9]. In this paper, we prove that all distance hereditary 2-connected graphs G such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

2. Recognition of distance hereditary graphs

Buckeley and Palka [6] and Bandelt and Mulder [2] independently obtained the following forbidden induced subgraphs characterization of distance-hereditary graphs. We will use this characterization and the following Theorem 2 for proving our desired class of graphs to be recognizable.

Theorem 1. *A graph G is distance-hereditary if and only if it contains no C_r, $r \geq 5$, nor any of the graphs in Fig. 1 as an induced subgraph.*

Theorem 2 (*Gupta et al. [10]*)**.** *Graphs G with $\text{diam}(G) = 2$ and graphs H with $\text{diam}(H) = \text{diam}(\overline{H}) = 3$ are recognizable.*

Lemma 3. *All distance hereditary 2-connected graphs G are recognizable.*

Proof. It is known [15] that all graphs with fewer than twelve vertices are reconstructible. We therefore assume that G has order n, where $n \geq 12$.

If H is one of the graphs in Fig. 1, or a cycle of order strictly less than n, then, using Kelly's lemma [5], we can determine whether H is an induced subgraph of G or not. Also, since graphs that are cycles are recognizable, we can determine whether C_n is an induced subgraph of G or not. Therefore, the proof of the lemma now follows by the fact that the connectivity of a graph is reconstructible. $\square$

3. Distance hereditary graphs G with $\text{diam}(G) = \text{diam}(\overline{G}) = 3$

A graph is *self-centred* if all its vertices have the same eccentricity. The *neighbourhood* of a vertex v in a graph G, denoted $N_G(v)$ or simply $N(v)$, is the set of all vertices adjacent to v in G. Lemmas 4 and 5 are used to prove Lemma 6, which will help us to know the structure of distance hereditary 2-connected graphs G with $\text{diam}(G)=\text{diam}(\overline{G})=3$.

Lemma 4. *If $\text{diam}(G) \geq 3$, then $\text{diam}(\overline{G}) \leq 3$.*