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Note

Independence-domination duality in weighted graphs

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ABSTRACT

Given a partition $\mathcal{V} = (V_1, \ldots, V_m)$ of the vertex set of a graph G, an *independent transversal* (IT) is an independent set in G that contains one vertex from each V_i. A *fractional IT* is a non-negative real valued function on $V(G)$ that represents each part with total weight at least 1, and belongs as a vector to the convex hull of the incidence vectors of independent sets in the graph. It is known that if the domination number of the graph induced on the union of every k parts V_i is at least k, then there is a fractional IT. We prove a weighted version of this result. This is a special case of a general conjecture, on the weighted version of a duality phenomenon, between independence and domination in pairs of graphs on the same vertex set.

1. Introduction

1.1. Domination and collective domination

All graphs in this paper are assumed to be simple, namely not containing parallel edges or loops. The (open) neighborhood of a vertex v in a graph G, denoted by $\tilde{N}(v) = \tilde{N}_G(v)$, is the set of all vertices connected to v. Given a set D of vertices we write $\tilde{N}(D)$ for $\bigcup_{v \in D} \tilde{N}(v)$. Let $N(D) = N_G(D) = \tilde{N}(D) \cup D$. A set D is said to be *dominating* if $N(D) = V$ and *totally dominating* if $\tilde{N}(D) = V$. The minimal size of a dominating set is denoted by $\gamma(G)$, and the minimal size of a totally dominating set by $\tilde{\gamma}(G)$.

There is a *collective* version of domination. Given a system of graphs $\mathcal{G} = (G_1, \ldots, G_k)$ on the same vertex set V, a system $\mathcal{D} = (D_1, \ldots, D_k)$ of subsets of V is said to be *collectively dominating* if $\bigcup_{i \le k} N_{G_i}(D_i) = V$. Let $\gamma_\cup(\mathcal{G})$ be the minimum of $\sum_{i \le k} |D_i|$ over all collectively dominating systems.

1.2. Independence and joint independence

A set of vertices is said to be *independent* in G if its elements are pairwise non-adjacent. The complex (closed down hypergraph) of independent sets in G is denoted by $\mathcal{I}(G)$. The *independence polytope* of G, denoted by $IP(G)$, is the convex hull of the characteristic vectors of the sets in $\mathcal{I}(G)$. For a system of graphs $\mathcal{G} = (G_1, \ldots, G_k)$ on V the *joint independence number*, $\alpha_\cap(\mathcal{G})$, is $\max\{|I| : I \in \cap_{i \le k} \mathcal{I}(G_i)\}$. The *fractional joint independence number*, $\alpha^*_\cap(\mathcal{G})$, is $\max\{\vec{x} \cdot \vec{1} : \vec{x} \in \bigcap_{i \le k} IP(G_i)\}$.

We shall mainly deal with the case $k = 2$. Let us first observe that it is possible to have $\alpha^*_\cap(G_1, G_2) < \min(\alpha(G_1), \alpha(G_2))$.

Example 1.1. Let G_1 be obtained from the complete bipartite graph with respective sides $\{v_1, \ldots, v_6\}$ and $\{u_1, u_2\}$, by the addition of the edges v_1v_2, v_3v_4 and u_1u_2, and let $G_2 = \bar{G_1}$, namely the complement of G_1. Then $\alpha(G_1) = \alpha(G_2) = 4$, while $\alpha^*_\cap(G_1, G_2) = 2$, the optimal vector in $IP(G_1) \cap IP(G_2)$ being the constant $\frac{1}{4}$ vector.

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A graph H is called a *partition graph* if it is the disjoint union of cliques. In a partition graph $\alpha = \gamma$. The union of two systems of disjoint cliques is the line graph of a bipartite graph, having the set of cliques in one system as one side of the graph, and the set of cliques in the other system as the other side, an edge connecting two vertices (namely, cliques in different systems) if they intersect. Thus, by König's famous duality theorem [4], we have the following:

Theorem 1.2 ([1]). *If G and H are partition graphs on the same vertex set, then*

$$\alpha_\cap(G, H) = \gamma_\cup(G, H).$$

There are graphs in which $\alpha > \gamma$, and thus equality does not necessarily hold for general pairs (G, H) of graphs, even when $G = H$. On the other hand, since a maximal independent set is dominating, we have $\gamma(G) \le \alpha(G)$ in every graph G. The corresponding inequality for pairs of graphs is not necessarily true, as the following example shows.

Example 1.3. Let $G = P_4$, namely the 4-vertex path, and let H be its complement. Then $\alpha_\cap(G, H) = 1$ and $\gamma_\cup(G, H) = 2$, so $\alpha_\cap(G, H) < \gamma_\cup(G, H)$.

However, as was shown in [1], if $\alpha_\cap$ is replaced by its fractional version, then the non-trivial inequality in Theorem 1.2 does hold.

Theorem 1.4 ([1]). *For any two graphs G and H on the same set of vertices we have*

$$\alpha^*_\cap(G, H) \ge \gamma_\cup(G, H).$$

In Example 1.3 $\vec{\frac{1}{2}} \in IP(G) \cap IP(H)$, and $\alpha^*_\cap(G, H) = 2$, so $\alpha^*_\cap(G, H) = \gamma_\cup(G, H)$.

1.3. Independent transversals

When one graph in the pair (G, H), say H, is a partition graph, the parameters $\alpha_\cap(G, H)$ and $\alpha^*_\cap(G, H)$ can be described using the terminology of so-called *independent transversals*. Given a graph G and a partition $\mathcal{V} = (V_1, \dots, V_m)$ of $V(G)$, an independent transversal (IT) is an independent set in G consisting of the choice of one vertex from each set V_i. A *partial IT* is an independent set representing some V_i's (so, it is the independent range of a partial choice function from $\mathcal{V}$). A function $f : V \to \mathbb{R}^+$ where $\mathbb{R}^+ = \{x \in \mathbb{R} | x \ge 0\}$ is called a *partial fractional IT* if, when viewed as a vector, it belongs to $IP(G)$, and $\sum_{v \in V_j} f(v) \le 1$ for all $j \le m$. If $\sum_{v \in V_j} f(v) = 1$ for all $j \le m$ then f is called a *fractional IT*. As we shall later see (Lemma 3.2) this means that $f \in IP(H) \cap IP(G)$ where H is the partition graph defined by the partition $\mathcal{V}$.

For $I \subseteq [m]$ let $V_I = \bigcup_{i \in I} V_i$.

The following was proved in [3]:

Theorem 1.5 ([3]). *If $\tilde{\gamma}(G[V_I]) \ge 2|I| - 1$ for every $I \subseteq [m]$ then there exists an IT.*

Theorem 1.4, applied to the case in which H is a partition graph, yields the following:

Theorem 1.6 ([1]). *If $\gamma(G[V_I]) \ge |I|$ for every $I \subseteq [m]$ then there exists a fractional IT.*

In this paper, we prove (Theorem 3.1) the vertex-weighted versions of this last theorem. This is the weighted version of the case of Theorem 1.4, in which H is the union of disjoint cliques. We conjecture that in fact the weighted version of Theorem 1.4 is true for all graphs H.

2. Weighted versions

In [2] a weighted version of Theorem 1.5 was proved. As is often the case with weighted versions, the motivation came from decompositions: weighted results give, by duality, fractional decompositions results. A conjecture first appearing in [2] (but having been folklore before) is that if $|V_i| \ge 2\Delta(G)$ then there exists a partition of $V(G)$ into $\max_{i \le m} |V_i|$ IT's. The weighted version of Theorem 1.5 yields the existence of a fractional such decomposition.

Notation 2.1. Given a real valued function f on a set S, and a set $A \subseteq S$, define $f[A] = \sum_{a \in A} f(a)$. We also write $|f| = f[S]$ and we call $|f|$ the *size* of f.

Remark 2.2. In this paper, $\mathbb{N} = \{a \in \mathbb{Z} | a \ge 0\}$

Definition 2.3. Let $G = (V, E)$ be a graph, and let $w : V \to \mathbb{N}$ be a weight function on V. We say that a function $f : V \to \mathbb{N}$ *w-dominates* a set U of vertices, if $f[N(u)] \ge w(u)$ for every $u \in U$. We say that f is *w-dominating* if it w-dominates V. The *weighted domination number* $\gamma^w(G)$ is $\min\{|f| \mid f \text{ is w-dominating}\}$.