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Note

An infinite family of subcubic graphs with unbounded packing chromatic number

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ABSTRACT

Recently, Balogh et al. (2018) answered in negative the question that was posed in several earlier papers whether the packing chromatic number is bounded in the class of graphs with maximum degree 3. In this note, we present an explicit infinite family of subcubic graphs with unbounded packing chromatic number.

Given a graph G, the *distance* between two vertices u and v in G, denoted by $d_G(u, v)$, is the length of a shortest u, v-path (we often drop the subscript if the graph G is clear from context). The maximum of $\{d_G(x, y) : x, y \in V(G)\}$ is called the *diameter* of G and denoted by $\mathrm{diam}(G)$. An *i-packing* in G, where i is a positive integer, is a subset W of the vertex set of G such that the distance between any two distinct vertices from W is greater than i. This concept generalizes the notion of an independent set, which is equivalent to a 1-packing. The *packing chromatic number* of G is the smallest integer k such that the vertex set of G can be partitioned into sets $V_1, \ldots, V_k$, where V_i is an i-packing for each $i \in \{1, \ldots, k\}$. This invariant is well defined in any graph G and is denoted by $\chi_\rho(G)$. The corresponding mapping $c : V(G) \longrightarrow \{1, \ldots, k\}$ having the property that $c(u) = c(v) = i$ implies $d_G(u, v) > i$ is called a *k-packing coloring*.

The concept of packing chromatic number of a graph was introduced a decade ago under the name broadcast chromatic number [12], and the current name was given in [4]. A number of authors have studied this invariant, cf. a selection of recent papers [1,3,5–8,10,11,13–17]. In particular, it was shown that the problem of determining the packing chromatic number is computationally (very) hard [9] as its decision version is NP-complete even when restricted to trees. Already in the seminal paper [12] it was observed that there is no upper bound for the packing chromatic number in the class of graphs with fixed maximum degree Δ when $\Delta \geq 4$, while the question for subcubic graphs (that is, the graphs with $\Delta \leq 3$) intrigued several authors, see [5,7,11,12]. In particular, a subcubic graph with packing chromatic number 13 was found in [11], and a subcubic graph with packing chromatic number 14 was constructed in [7], but no subcubic graph with bigger packing chromatic number was known. Finally, in [2] the authors proved that the packing chromatic number of subcubic graphs is unbounded. The proof is rather involved and uses the so-called configuration model technique. However, this remarkable proof does not give an explicit construction of a family of subcubic graphs with unbounded packing chromatic number.

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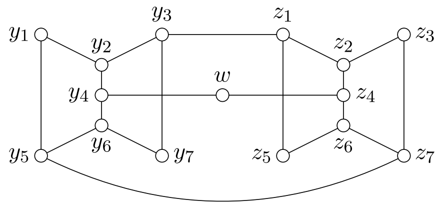

Fig. 1. Graph H.

In this note we present a family of subcubic graphs G_k with the property that $\chi_\rho(G_k) \geq 2k+9$. The main tool in the proof is to keep the diameter of the graphs in the family under control (that is, $\mathrm{diam}(G_k) \leq 2k+6$), and at the same time a packing coloring of these graphs requires more colors than the diameter. We are able to compute the bounds for the packing chromatic numbers of the graphs G_k by using recursive structure of the family G_k (each graph G_k contains two copies of G_{k-1} as induced subgraphs).

In the remainder of this note, we present the construction and prove the mentioned bounds for the diameter and the packing chromatic number of the graphs G_k. The basic building block in the construction is the graph H in Fig. 1. Note that $\mathrm{diam}(H) = 4$.

Lemma 1. *The packing chromatic number of the graph H, shown in Fig. 1, is at least* 7.

Proof. Suppose to the contrary that $\chi_\rho(H) \leq 6$ and let c be an arbitrary 6-packing coloring of the graph H. Denote by Y the subgraph of H induced by the vertices $y_1, \ldots, y_7$, and by Z the subgraph of H induced by the vertices $z_1, \ldots, z_7$. For c restricted to Y (and analogously to Z) we have $|c^{-1}(1) \cap V(Y)| \leq 3$, $|c^{-1}(2) \cap V(Y)| \leq 2$ and $|c^{-1}(3) \cap V(Y)| \leq 1$. Therefore, $|c^{-1}(1) \cap V(H)| \leq 7$, $|c^{-1}(2) \cap V(H)| \leq 5$, $|c^{-1}(3) \cap V(H)| \leq 2$. Since the diameter of H is 4, we also have $|c^{-1}(l) \cap V(H)| = 1$ for any $l \in \{4, 5, 6\}$. We distinguish four cases with respect to $|c^{-1}(2) \cap V(H)|$.

Case 1. $|c^{-1}(2) \cap V(H)| = 5$.
All vertices, which get color 2, are uniquely determined and these are y_1, y_7, w, z_3 and z_5. Since $c(w) \neq 1$, we have $|c^{-1}(1) \cap V(H)| \leq 6$. Recall that $|c^{-1}(1) \cap V(Y)| \leq 3$ and $|c^{-1}(1) \cap V(Z)| \leq 3$, but it is easy to see that if $|c^{-1}(1) \cap V(Y)| = 3$ then $|c^{-1}(1) \cap V(Z)|$ cannot reach the established upper bound (respectively, if $|c^{-1}(1) \cap V(Z)| = 3$, then $|c^{-1}(1) \cap V(Y)| < 3$). This yields $|c^{-1}(1) \cap V(H)| \leq 5$. If $|c^{-1}(1) \cap V(H)| = 5$ (and $|c^{-1}(2) \cap V(H)| = 5$), then vertices colored by 1 are y_3, y_4, y_5, z_2 and z_6 (or z_1, z_4, z_7, y_2 and y_6), but then $|c^{-1}(3) \cap V(H)| \leq 1$ and hence $\sum_{i=1}^{6} |c^{-1}(i) \cap V(H)| \leq 14$. This is contradiction since H has 15 vertices, but we can color only 14 of them. The same contradiction arises if $|c^{-1}(1) \cap V(H)| < 5$.

Case 2. $|c^{-1}(2) \cap V(H)| = 4$.
Suppose $c(w) \neq 2$. Then the vertices, which are colored by 2, are y_1, y_7, z_3 and z_5. If $c(w) = 1$, then $|c^{-1}(1) \cap V(H)| \leq 5$ and $\sum_{i=1}^{6} |c^{-1}(i) \cap V(H)| \leq 14$, therefore we get the same contradiction as above. If $c(w) \neq 1$, then we have the analogous situation as in Case 1, which implies $|c^{-1}(1) \cap V(H)| \leq 5$ and $\sum_{i=1}^{6} |c^{-1}(i) \cap V(H)| \leq 14$, a contradiction.
Next, suppose $c(w) = 2$. Without loss of generality we may assume that $|c^{-1}(2) \cap V(Y)| = 2$ (and $|c^{-1}(2) \cap V(Z)| = 1$). This is not possible if $c(y_3) = 2$ and $c(y_5) = 2$, hence the vertices of Y colored by 2 are y_1 and y_7. It is clear that $|c^{-1}(1) \cap V(H)| \leq 6$, but it is also easy to see that if $|c^{-1}(1) \cap V(Y)| = 3$, then $|c^{-1}(1) \cap V(Z)| \leq 2$ (note that $|c^{-1}(2) \cap V(Z)| = 1$, namely $c(z_3) = 2$ or $c(z_5) = 2$). Hence $|c^{-1}(1) \cap V(H)| \leq 5$ and we get the same contradiction as above.

Case 3. $|c^{-1}(2) \cap V(H)| = 3$.
If $c(w) \neq 1$, then $|c^{-1}(1) \cap V(H)| \leq 6$ and $\sum_{i=1}^{6} |c^{-1}(i) \cap V(H)| \leq 14$, a contradiction.
Suppose $c(w) = 1$. Again, without loss of generality we may assume that $|c^{-1}(2) \cap V(Y)| = 2$ (and $|c^{-1}(2) \cap V(Z)| = 1$). The vertices of Y, colored by 2, are either y_1 and y_7 or y_3 and y_5, but in each case this yields $|c^{-1}(1) \cap V(Y)| \leq 2$ (note that $c(y_4) \neq 1$). Hence $|c^{-1}(1) \cap V(H)| \leq 6$ and $\sum_{i=1}^{6} |c^{-1}(i) \cap V(H)| \leq 14$, which is a contradiction.

Case 4. $|c^{-1}(2) \cap V(H)| \leq 2$.
In this case we have $\sum_{i=1}^{6} |c^{-1}(i) \cap V(H)| \leq 14$, which is again a contradiction. □

Next, starting from two copies of graph H (denoted by H' and H''), adding five vertices, and adding edges as shown in Fig. 2 we obtain graph G_0.

Lemma 2. *The diameter of the graph G_0, shown in Fig. 2, is at most* 6.

Proof. Recall that the diameter of the graph H is 4. Hence it is clear that the distance between any two vertices of the subgraph H' (or H'') of G_0 is at most 4. Therefore, we only need to check the distances between any two vertices of G_0, one of which is from $V(H') \cup \{a, b, c, d, x\}$ and the other from $V(H'') \cup \{a, b, c, d, x\}$.