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Note

A lower bound on the acyclic matching number of subcubic graphs

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ABSTRACT

The acyclic matching number of a graph G is the largest size of an acyclic matching in G, that is, a matching M in G such that the subgraph of G induced by the vertices incident to edges in M is a forest. We show that the acyclic matching number of a connected subcubic graph G with m edges is at least $m/6$ except for two graphs of order 5 and 6.

1. Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation. A matching M in a graph G is *acyclic* [7] if the subgraph of G induced by the set of vertices that are incident to some edge in M is a forest, and the *acyclic matching number* $\nu_{ac}(G)$ of G is the maximum size of an acyclic matching in G. While the ordinary matching number $\nu(G)$ of G is tractable [4], it has been known for some time that the acyclic matching number is NP-hard for graphs of maximum degree 5 [7,15]. Recently, we [6] showed that just deciding the equality of $\nu(G)$ and $\nu_{ac}(G)$ is already NP-complete when restricted to bipartite graphs G of maximum degree 4. The complexity of the acyclic matching number for cubic graphs is unknown.

In the present paper we establish a lower bound on the acyclic matching number of subcubic graphs. Similar results were obtained for the matching number [2,8,10,14], and also for the induced matching number [11–13]. Baste and Rautenbach [1] showed that the edge set of a graph G of maximum degree $\Delta(G)$ can be partitioned into at most $\Delta(G)^2$ acyclic matchings in G. This implies $\nu_{ac}(G) \geq m(G)/\Delta(G)^2$, where $m(G)$ denotes the size of G. For subcubic graphs, this simplifies to $\nu_{ac}(G) \geq m(G)/9$. This latter bound also follows from a lower bound [13] on the induced matching number, which is always at most the acyclic matching number. While the bound is tight for $K_{3,3}$, excluding some small graphs allows a considerable improvement. Let K_4^+ be the graph that arises by subdividing one edge of K_4 once.

We prove the following.

Theorem 1. *If G is a connected subcubic graph that is not isomorphic to K_4^+ or $K_{3,3}$, then $\nu_{ac}(G) \geq m(G)/6$.*

Since every subcubic graph G of order $n(G)$ satisfies $m(G) \leq 3n(G)/2$, Theorem 1 is an immediate consequence of the following stronger result. For two graphs G and H, let $\kappa_G(H)$ denote the number of components of G that are isomorphic to H.

Theorem 2. *If G is a subcubic graph without isolated vertices, then*

$$\nu_{ac}(G) \geq \frac{1}{4}\left(n(G) - \kappa_G(K_{2,3}) - \kappa_G(K_4^+) - 2\kappa_G(K_{3,3})\right).$$

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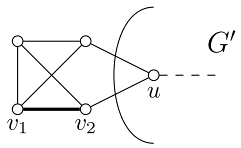

Fig. 1. An illustration of Claim 1.

Note that Theorem 2 is tight; examples are K_4, $K_{2,2}$, $K_{1,3}$, or the graph obtained from $K_{1,3}$ by replacing each endvertex with an endblock isomorphic to $K_{2,3}$. The proof of Theorem 2 is postponed to the second section. The reduction arguments within that proof easily lead to a polynomial time algorithm computing acyclic matchings of the guaranteed size.

In the third section, we conclude with some open problems.

2. Proof of Theorem 2

The proof is by contradiction. Therefore, suppose that G is a counterexample to Theorem 2 that is of minimum order n. A graph is *special* if it is isomorphic to $K_{2,3}$, K_4^+, or $K_{3,3}$. Clearly, G is connected, not special, and n is at least 5. By our initial assumption, $\nu_{ac}(G) < n/4$.

We derive a contradiction using a series of claims.

Claim 1. *No subgraph of G is isomorphic to K_4^+.*

Proof of Claim 1. Suppose that G has a subgraph H that is isomorphic to K_4^+. Let v_1, v_2, v_3, and v_4 be the vertices of degree 3 in H, and let u the vertex of degree 2 in H. Let $G' = G - \{v_1, v_2, v_3, v_4\}$, see Fig. 1.

Since G is connected, the graph G' is connected. Since u has degree 1 in G', the graph G' is not special. By the choice of G, the graph G' is no counterexample to Theorem 2, and, hence, it has an acyclic matching M' of size at least $n(G')/4 = n/4 - 1$. Adding the edge v_1v_2 to M' yields an acyclic matching in G of size at least $n/4$, which is a contradiction. □

Claim 2. *No endblock of G is isomorphic to $K_{2,3}$.*

Proof of Claim 2. Suppose that some endblock B of G is isomorphic to $K_{2,3}$. Let u be the unique cutvertex of G in B. Clearly, the vertex u has degree 2 in B. The graph $G' = G - (V(B) \setminus \{u\})$ is connected, and, since u has degree 1 in G', it is not special. Therefore, by the choice of G, the graph G' has an acyclic matching M' of size at least $n(G')/4 = n/4 - 1$. Adding an edge of B that is not incident to u to M' yields an acyclic matching in G of size at least $n/4$, which is a contradiction. □

Claim 3. *No two vertices of degree* 1 *have a common neighbor.*

Proof of Claim 3. Suppose that u and v are two vertices of degree 1, and that w is their common neighbor. Let $G' = G - \{u, v, w\}$. Since G' is connected and not isomorphic to $K_{3,3}$, the choice of G implies that G' has an acyclic matching M' of size at least $(n(G') - 1)/4 = n/4 - 1$. Since w does not lie on any cycle in G, adding the edge uw to M' yields an acyclic matching in G of size at least $n/4$, which is a contradiction. □

Claim 4. *No vertex of degree* 1 *is adjacent to a vertex that does not lie on a cycle.*

Proof of Claim 4. Suppose that u is a vertex of degree 1 that is adjacent to a vertex v that does not lie on a cycle. By Claim 3, the graph $G' = G - \{u, v\}$ has no isolated vertex. Since G' has at most two components, and no component of G' is isomorphic to $K_{3,3}$, the choice of G implies that G' has an acyclic matching M' of size at least $(n(G') - 2)/4 = n/4 - 1$. Since v does not lie on a cycle, adding the edge uv to M' yields an acyclic matching in G of size at least $n/4$, which is a contradiction. □

Claim 5. *The minimum degree of G is at least* 2.

Proof of Claim 5. Suppose that u is a vertex of degree 1. By Claim 4, the neighbor v of u lies on a cycle C in G. Let x and w be the neighbors of v on C.

First, suppose that w has no neighbor of degree 1.

If $G - \{u, v, w\}$ contains an isolated vertex, then this is necessarily the vertex x, and $N_G(x) = \{v, w\}$. In this case, let $G' = G - \{u, v, w, x\}$, see the left of Fig. 2.

Clearly, the graph G' is connected and not isomorphic to $K_{3,3}$. If G' is isomorphic to K_4^+ or $K_{2,3}$, then it follows easily that $\nu_{ac}(G) \geq 3 > 9/4 = n/4$, which is a contradiction. Hence, G' is not special, which implies that G' has an acyclic matching M' of size at least $n(G')/4 = n/4 - 1$. Adding the edge uv to M' yields an acyclic matching in G of size at least $n/4$, which is