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On higher indicators of regular association schemes

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A R T I C L E I N F O

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A B S T R A C T

In the present paper, we will define the higher Frobenius–Schur indicators and the higher indicators of association schemes as a generalization of those of finite groups. The higher indicators of any association scheme are always positive rational numbers. Especially, for any positive integer n, the nth indicator of any regular association scheme is the number of relations such that its strong girth divides n. Thus, all higher indicators of any regular association scheme are natural numbers, and the sequence of the indicators is periodic. We will show that the converses of these facts are also true for finite exponent association schemes. Finally, we introduce a family of infinite exponent association schemes all higher indicators of which are natural numbers and the sequence of the indicators of which is periodic.

1. Introduction

First, we recall higher Frobenius–Schur indicators and higher indicators of finite groups (see [3,4] and [5] for more details). For a finite group G and any complex irreducible character χ of G, we define the nth Frobenius–Schur indicator $\nu_n(\chi)$ for any positive integer n by

$$\nu_n(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^n).$$

Moreover, we define the nth indicator $\nu_n(G)$ of G for any positive integer n by

$$\nu_n(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\nu_n(\chi),$$

where $\mathrm{Irr}(G)$ means the set of all irreducible characters of G. It is well known that $\nu_n(G) = \#\{g \in G \mid g^n = 1\}$, namely $\nu_n(G)$ is equal to the number of elements whose order divides n. Especially, $\nu_n(G) \in \mathbb{N}$ for all $n \geq 1$ and the sequence $\{\nu_n(G)\}_{n \geq 1}$ of the indicators of G is periodic with period $\exp(G)$, where $\exp(G)$ means the exponent of the group G.

Let S be an association scheme, and let $\mathrm{Irr}(S)$ be the set of all irreducible characters of the complex adjacency algebra $\mathbb{C}S$ of S. In [2], Higman generalized the notion of the second Frobenius–Schur indicator for any irreducible character of the complex adjacency algebra of an association scheme, namely

$$\nu_2(\chi) = \frac{m_\chi}{n_S \chi(\sigma_1)} \sum_{s \in S} \frac{1}{n_s} \chi({\sigma_s}^2).$$

He also showed that the second indicator $\nu_2(S) := \sum_{\chi \in \mathrm{Irr}(S)} \chi(\sigma_1)\nu_2(\chi)$ is equal to the number of symmetric relations of S.

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In the present paper, we define in Section 3 the nth Frobenius–Schur indicator $\nu_n(\chi)$ for any irreducible character $\chi \in \mathrm{Irr}(S)$ and any positive integer n by

$$\nu_n(\chi) := \frac{m_\chi}{n_S\chi(\sigma_1)} \sum_{s\in S} \frac{1}{{n_s}^{n-1}} \chi({\sigma_s}^n),$$

and the nth indicator $\nu_n(S)$ of S by

$$\nu_n(S) := \sum_{\chi\in\mathrm{Irr}(S)} \chi(\sigma_1)\nu_n(\chi).$$

By direct calculation, it holds that for $n > 1$,

$$\nu_n(S) = \sum_{s\in S} \frac{1}{{n_s}^{n-1}} a_{s^n 1},$$

and $\nu_1(S) = 1$. Thus, the nth indicator $\nu_n(S)$ is a positive rational number for all n. Unlike the case of the higher indicators of finite groups, there exist many association schemes which nth indicator is not an integer for some n (for example, rank 2 association schemes of order $|X| \geq 3$). On the other hand, if an association scheme S is regular, we have that for any $n \in \mathbb{N}$,

$$\nu_n(S) = \#\{s \in S \mid \mathrm{sg}(s) \mid n\},$$

where $\mathrm{sg}(s)$ means the strong girth of a relation s (Theorem 8). Thus, $\nu_n(S) \in \mathbb{N}$ for all $n \geq 1$ and the sequence $\{\nu_n(S)\}_{n\geq 1}$ is periodic with period $\exp(S)$.

Any regular association scheme is of finite exponent. However, finite exponent association schemes are not necessarily regular. We can characterize regular association schemes among finite exponent association schemes in terms of higher indicators. We will show in Theorem 10 that for a finite exponent association scheme S, the following are equivalent:

1. S is regular,
2. $\{\nu_n(S)\}_{n\geq 1}$ is periodic,
3. $\nu_n(S) \in \mathbb{N}$ for all $n \geq 1$.

In the last section, we construct infinite exponent association schemes all higher indicators of which are natural numbers, and the sequence of indicators of which is periodic. Since infinite exponent association schemes are not regular, we cannot remove the condition of finite exponent in Theorem 10.

2. Preliminaries

2.1. Association schemes

As far as scheme theoretic terminology and notation is concerned we refer to [7]. Let X be a finite set. We write 1 to denote the set of all pairs (x, x) with $x \in X$. For each subset s of the Cartesian product $X \times X$, we define s^* to be the set of all pairs (y, z) with $(z, y) \in s$. Whenever x stands for an element in X and s for a subset of $X \times X$, we define xs to be the set of all elements $y \in X$ such that $(x, y) \in s$.

Let S be a partition of $X \times X$ with $1 \in S$, and assume that, for each relation $s \in S$, we have $s^* \in S$. The set S is called an *association scheme* or simply a *scheme on* X if, for any three relations s, t, and $u \in S$, there exists an integer a_{stu} such that, for any two elements $y \in X$ and $z \in yu$, $|ys \cap zt^*| = a_{stu}$. For each relation $s \in S$, the integer $n_s := a_{ss^*1}$ is called the *valency* of s. For a subset T of S, we define $n_T := \sum_{t\in T} n_t$. A relation $s \in S$ is called *symmetric* if $s^* = s$.

Let s be a relation in S, let n be an integer with $n \geq 3$, and let $s_1, \ldots, s_n$ be relations in S. We inductively define an integer $a_{s_1\ldots s_n s}$ by

$$a_{s_1\ldots s_n s} := \sum_{\ell\in S} a_{s_1\ldots s_{n-1}\ell}\, a_{\ell s_n s}.$$

For any relations $s, \ell, \ell' \in S$ and $n \geq 2$, we set $a_{s^n\ell} = a_{\underbrace{s\ldots s}_{n}\ell}$ and $a_{\ell s^n \ell'} = a_{\ell\underbrace{s\ldots s}_{n}\ell'}$.

Let T and U be nonempty subsets of S. We define TU to be the set of all relations $s \in S$ such that there exist relations $t \in T$ and $u \in U$ with $a_{tus} \neq 0$. The set TU is called the *complex product of* T *and* U. Whenever s is a relation in S and T is a nonempty subset of S, we write sT instead of $\{s\}T$ and Ts instead of $T\{s\}$.

For each relation $s \in S$, we define the $\{0, 1\}$-matrix σ_s whose rows and columns are indexed by the elements of X by

$$(\sigma_s)_{xy} := \begin{cases} 1 & \text{if } (x, y) \in s, \\ 0 & \text{if } (x, y) \notin s. \end{cases}$$

For a subset T of S, we define $\sigma_T := \sum_{t\in T}\sigma_t$. We define the *adjacency algebra* $\mathbb{C}S$ *of* S *over* the complex number field $\mathbb{C}$ by $\mathbb{C}S := \oplus_{s\in S}\mathbb{C}\sigma_s$ as a matrix algebra.