Discrete Mathematics 341 (2018) 2049–2057

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A study on oriented relative clique number

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ARTICLE INFO

Article history:
Received 2 May 2017
Received in revised form 14 March 2018
Accepted 2 April 2018
Available online 30 April 2018

Keywords:
Oriented coloring
Oriented cliques
Maximum degree
Outerplanar graphs
Planar graphs

ABSTRACT

An oriented graph is a directed graph with no directed cycle of length one or two. The relative clique number of an oriented graph is the cardinality of a largest subset X of vertices such that each pair of vertices is either adjacent or connected by a directed 2-path. It is known that the oriented relative clique number of a planar graph is at most 80. Here we improve the upper bound to 32. We also prove an upper bound of 14 for oriented relative clique number of triangle-free planar graphs. Furthermore, we determine the exact values of oriented relative clique number for the families of outerplanar graphs with girth at least g and planar graphs with girth at least $g+2$ for all $g \geq 3$. Moreover, we study the relation of oriented relative clique number with oriented chromatic number, oriented absolute clique number and maximum degree of a graph. We also show that oriented relative clique number of a connected subcubic graph is at most seven which weakly supports a conjecture by Sopena (JGT 1997).

1. Introduction

An *oriented graph* $\overrightarrow{G}$ is a directed graph having no directed cycle of length one or two with set of vertices $V(\overrightarrow{G})$ and set of arcs $A(\overrightarrow{G})$. We denote by G the underlying graph of $\overrightarrow{G}$ and $\overrightarrow{G}$ is an orientation of G. An oriented graph with three vertices u, v, w and two arcs $\overrightarrow{uv}$ and $\overrightarrow{vw}$ is a *directed* 2*-path* or a 2*-dipath* where u and w are *terminal* vertices and v is an *internal* vertex.

Colorings of oriented graphs first appeared in the work of Courcelle [1] on the monadic second order logic of graphs. Since then it has been considered by many researchers, following the work of Raspaud and Sopena [12] on oriented colorings of planar graphs.

An *oriented k-coloring* [14] of an oriented graph $\overrightarrow{G}$ is a mapping ϕ from the vertex set $V(\overrightarrow{G})$ to a set of k colors such that, (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent and (ii) for two arcs $\overrightarrow{uv}$ and $\overrightarrow{wx}$ of $\overrightarrow{G}$, $\phi(u) = \phi(x)$ implies $\phi(v) \neq \phi(w)$. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the smallest integer k for which $\overrightarrow{G}$ has an oriented k-coloring.

Notice that the terminal vertices of a 2-dipath must receive distinct colors in every oriented coloring. In fact, for providing an oriented coloring of an oriented graph, only the pairs of vertices which are either adjacent or connected by a 2-dipath must receive distinct colors (that is, for every other type of pair of vertices there exists an oriented coloring which assigns the same color to the pair of vertices).

An *oriented relative clique* [10] of an oriented graph $\overrightarrow{G}$ is a set $R \subseteq V(\overrightarrow{G})$ of vertices such that any two distinct vertices from R are at directed distance at most 2, that is, either adjacent or connected by a 2-dipath, in $\overrightarrow{G}$. Note that the internal vertex of a 2-dipath connecting a pair of vertices of R may not be contained in R. The *oriented relative clique number* $\omega_{ro}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the maximum order of an oriented relative clique of $\overrightarrow{G}$.

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https://doi.org/10.1016/j.disc.2018.04.001
0012-365X/

An *oriented absolute clique* or *oclique* [5] is an oriented graph $\overrightarrow{G}$ for which $\chi_o(\overrightarrow{G}) = |V(\overrightarrow{G})|$. Ocliques can be characterized as those oriented graphs whose any two distinct vertices are at directed distance at most 2 [5]. The *oriented absolute clique number* $\omega_{ao}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the maximum order of an oclique contained in $\overrightarrow{G}$ as a subgraph.

The oriented chromatic number $\chi_o(G)$ of a simple graph G is the maximum of the oriented chromatic numbers of all the oriented graphs with underlying graph G. The oriented chromatic number $\chi_o(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the oriented chromatic numbers of the graphs from the family $\mathcal{F}$. Oriented relative and absolute clique numbers for a simple graph and a family of graphs are defined similarly.

One of the long standing open questions in the domain of oriented coloring is the analog to the Four-Color Theorem, that is, what is the oriented chromatic number $\chi_o(\mathcal{P}_3)$ of the family of planar graphs $\mathcal{P}_3$? It is known that $18 \leq \chi_o(\mathcal{P}_3) \leq 80$ [8,12] and it is a popular belief that the actual value of $\chi_o(\mathcal{P}_3)$ is much less than 80. As the upper bound remains the best known bound for the last twenty years, it is natural to consider "easier" parameters. Clearly, from the definitions, $\omega_{ao}(G) \leq \omega_{ro}(G) \leq \chi_o(G)$ for any (oriented) graph G. Klostermeyer and MacGillivray [5] introduced the parameter ω_{ao} and showed that $\omega_{ao}(\mathcal{P}_3) \leq 36$. They also conjectured that $\omega_{ao}(\mathcal{P}_3) = 15$ which was positively settled [10]. The bounds for oriented relative clique number of planar graphs that follow easily from the relations between the parameters are $15 \leq \omega_{ro}(\mathcal{P}_3) \leq 80$ [14]. The natural question that follows is whether we can improve the bounds. Even though the parameter ω_{ro} is introduced in a recent work involving the third author [10] it has been discussed within the oriented graph community for some time now. In fact, the specific question whether one can improve the upper bound of $\omega_{ro}(\mathcal{P}_3) \leq 80$ was posed as an open problem in one of the open problem sessions in SGT 2013 (Oléron, France). In a recent survey on oriented coloring [14] the parameter oriented relative clique number is mentioned. In the same survey [14], Sopena asked to determine the largest possible values of the oriented relative clique numbers of planar and triangle-free planar graphs. We address both these questions and prove upper bounds for the corresponding numbers.

In this article,[1] we fix the notations and terminologies in Section 2. In Section 3 we study the relation among the three parameters, χ_o, ω_{ro} and ω_{ao} and show that determining ω_{ro} is an NP-hard problem even for the family of oriented bipartite graphs. In Section 4 we prove good lower and upper bounds of oriented relative clique number for the family of graphs with bounded maximum degree. We also show that the oriented relative clique number for the family of subcubic graphs is seven which weakly supports a conjecture by Sopena [13]. In Sections 5 and 6 we study the oriented relative clique number for the family of outerplanar graphs and planar graphs, respectively. In particular, we show $\omega_{ro}(\mathcal{P}_3) \leq 32$ and $\omega_{ro}(\mathcal{P}_4) \leq 14$, where $\mathcal{P}_4$ denotes the family of triangle free planar graphs. Both the results improve the upper bounds proved in the conference version [2] of this article. Furthermore, we determine the exact values of oriented relative clique number for the families of outerplanar graphs with girth at least g and planar graphs with girth at least $g+2$ for all $g \geq 3$. Finally, in Section 7 we conclude our work and propose some future directions of research on this topic.

2. Preliminaries

A *clique* C of a graph G is a subset of $V(G)$ such that any pair of vertices from C is adjacent. The *clique number* $\omega(G)$ of a graph G is the cardinality of a largest clique of G.

The set of all adjacent vertices of a vertex v in an (oriented) graph G is called its set of *neighbors* and is denoted by $N(v)$. The *closed neighborhood* of v is defined by $N[v] = N(v) \cup \{v\}$. Also $d(v) = |N(v)|$ denotes the *degree* of v. Let uv be an arc in $\overrightarrow{G}$. Then u is an *in-neighbor* of v and v is an *out-neighbor* of u. The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N^-(v)$ and $N^+(v)$, respectively.

Two vertices u and v *agree* on a third vertex w if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+, -\}$ and they *disagree* on w if $w \in N^\alpha(u) \cap N^\beta(v)$ for $\{\alpha, \beta\} = \{+, -\}$. A vertex v *sees* a vertex u if they are either adjacent or connected by a 2-dipath. If u and v are connected by a 2-dipath with internal vertex w, then we say that v sees u *through* w (or equivalently, u sees v *through* w).

Next we will define a partial order $\prec$ for oriented graphs. We have $\overrightarrow{G_1} \prec \overrightarrow{G_2}$ if either of the following conditions holds.

(i) $|V(\overrightarrow{G_1})| < |V(\overrightarrow{G_2})|$,

(ii) $|V(\overrightarrow{G_1})| = |V(\overrightarrow{G_2})|$ and $|A(\overrightarrow{G_1})| < |A(\overrightarrow{G_2})|$.

For our proofs, given an oriented graph $\overrightarrow{G}$ we will usually denote by R, a relative clique of cardinality $\omega_{ro}(\overrightarrow{G})$. Moreover, we denote by S the set $V(\overrightarrow{G}) \setminus R$ and for convenience, we call the vertices of R and S as *good vertices* and *helper vertices*, respectively.

Let $\overrightarrow{G}$ be an oriented graph. Then $\overrightarrow{G}^2$ is the graph with set of vertices $V(\overrightarrow{G}^2) = V(\overrightarrow{G})$ and set of edges $E(\overrightarrow{G}^2) = \{uv | u \text{ sees } v \text{ in } \overrightarrow{G}\}$. Note that $\overrightarrow{G}^2$ is an undirected graph.

An *isomorphism* of $\overrightarrow{G}$ to $\overrightarrow{H}$ is a bijective homomorphism whose inverse is also a homomorphism. An isomorphism of $\overrightarrow{G}$ to itself is an *automorphism*. An oriented graph $\overrightarrow{G}$ is *vertex transitive* if for any two vertices u, v of $\overrightarrow{G}$ there exists an automorphism f of $\overrightarrow{G}$ such that $f(u) = v$.

[1] A preliminary version of this article appeared in LAGOS 2015 [2].