Discrete Mathematics 341 (2018) 2058–2067

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Splitting a planar graph of girth 5 into two forests with trees of small diameter

Aleksey N. Glebov

Sobolev Institute of Mathematics SB RAS, 630090, Prospekt Ak. Koptyuga, 4, Novosibirsk, Russia

ARTICLE INFO

Article history:
Received 22 November 2016
Received in revised form 9 April 2018
Accepted 10 April 2018
Available online 30 April 2018

Keywords:
Planar graph
Path partition
P_k-free colouring
List colouring
Forest

ABSTRACT

In 1985, Mihok and recently Axenovich, Ueckerdt, and Weiner asked about the minimum integer $g^* > 3$ such that every planar graph with girth at least g^* admits a 2-colouring of its vertices where the length of every monochromatic path is bounded from above by a constant. By results of Glebov and Zambalaeva and of Axenovich et al., it follows that $5 \leq g^* \leq 6$. In this paper we establish that $g^* = 5$. Moreover, we prove that every planar graph of girth at least 5 admits a 2-colouring of its vertices such that every monochromatic component is a tree of diameter at most 6. We also present the list version of our result.

1. Introduction

In this paper we investigate the problem of splitting the vertex set of a planar graph $G = (V, E)$ into two subsets such that subgraphs induced by both subsets do not contain simple paths on k vertices, denoted P_k. The problem is equivalent to the vertex colouring of G by two colours such that G does not contain a monochromatic path P_k.

A graph G is *planar* if it can be drawn in the plane with no crossings. A *plane* graph is a planar drawing of a planar graph G. The *girth* of a graph G, denoted $g(G)$, is the length of its shortest cycle. An *m-colouring* of G is any colouring $c : V \to \{1, 2, \ldots, m\}$ of its vertices by m colours. A colouring c is *proper* if any two adjacent vertices have different colours. A graph G is (properly) *m-colourable* if there exists a proper m-colouring of its vertices. A colouring c is P_k*-free* if G contains no monochromatic path P_k. We say that a P_k-free colouring is *acyclic* if G contains no monochromatic cycle, and hence each monochromatic component of G is a tree of diameter at most $k - 2$.

Another important and intensively studied concept is *defective* $(d_1, \ldots, d_m)$-colouring of a graph G where vertices of every colour $i \in \{1, \ldots, m\}$ induce a subgraph of maximum degree at most d_i. Observe that both defective $(0, \ldots, 0)$-colouring and P_2-free m-colouring are identical to a proper m-colouring while a P_3-free colouring is equivalent to a defective $(1, \ldots, 1)$-colouring (where every monochromatic component is a vertex or an edge).

For all colourings introduced above it is interesting to consider their list versions. Suppose L is a list assignment for a graph G, which assigns a list of available colours $L(v)$ to every vertex $v \in V$. An *L-colouring* of G is a colouring $c : V \to \bigcup_{v\in V} L(v)$ such that $c(v) \in L(v)$ for every $v \in V$. A graph G is *m-choosable* if it admits a proper L-colouring for every list assignment L such that $|L(v)| = m$ for all $v \in V$. Along with proper L-colourings, we can consider (acyclic) P_k-free and defective L-colourings (in the case $d_1 = d_2 = \cdots = d_m = d$) and corresponding (acyclic) P_k-free m-choosable and defective $(d, d, \ldots, d)$-choosable graphs.

The notion of a P_k-free colouring was introduced by Chartrand, Geller, and Hedetniemi [11], who showed that for any $k > 0$ there exist planar graphs that do not admit a P_k-free 3-colouring. However, by the famous Four Colour Theorem [2,3],

E-mail address: angle@math.nsc.ru.

https://doi.org/10.1016/j.disc.2018.04.007
0012-365X/

every planar graph is 4-colourable or, equivalently, P_2-free 4-colourable. Cowen, Cowen, and Woodall [13] proved that every planar graph is defectively $(2, 2, 2)$-colourable. The well-known Grötzsch Theorem [16] yields that every triangle-free planar graph is P_2-free 3-colourable.

Recently, Axenovich, Ueckerdt, and Weiner [4] showed that for any $k > 0$ there exist triangle-free planar graphs (of girth 4) that are not P_k-free 2-colourable. Montassier and Ochem [21], for all $k > 0$, presented examples of 2-degenerate planar graphs of girth 4 (respectively, 5 and 7) that are not defectively (k, k)-colourable (respectively, not $(3, 1)$-colourable and not $(2, 0)$-colourable). Borodin et al. [8], for every $k > 0$, constructed 2-degenerate planar graphs of girth 6 that are not defectively $(k, 0)$-colourable.

On the other hand, it is known that if the *maximum average degree*, $mad(G) = \max_{H \subseteq G} 2\frac{|E(H)|}{|V(H)|}$, of a graph G is low, then G is defectively (d_1, d_2)-colourable for small constants d_1 and d_2. Borodin, Kostochka, and Yancey [10] proved that every graph G with $mad(G) \leq \frac{14}{5}$ is defectively $(1, 1)$-colourable or, equivalently, P_3-free 2-colourable. Borodin and Ivanova [7] proved that every graph G with $g(G) \geq 7$ and $mad(G) < \frac{14}{5}$ admits a list 2-colouring where every monochromatic component is a path with at most three vertices. Since every planar graph G with girth at least g has $mad(G) < \frac{2g}{g-2}$, the results in [7,10] are valid for planar graphs of girth at least 7. Kim, Kostochka, and Zhu [19] proved that every triangle-free graph G with $|E(H)| < \frac{11|V(H)|+5}{9}$ for every subgraph $H \subseteq G$ admits a defective $(0, 1)$-colouring. This implies that every planar graph with girth at least 11 is defectively $(0, 1)$-colourable. By the results of Borodin and Kostochka [9], it follows that every planar graph G with $g(G) \geq 8$ (respectively, $g(G) \geq 7$, $g(G) \geq 6$, and $g(G) \geq 5$) is defectively $(0, 2)$-colourable (respectively, $(0, 4)$-colourable, $(1, 4)$-colourable, and $(2, 6)$-colourable). Choi and Raspaud [12] proved that planar graphs with girth at least 5 are defectively $(3, 5)$-colourable. Havet and Sereni [17] obtained, for every $k \geq 0$, that every graph G with $mad(G) < \frac{4k+4}{k+2}$ is defectively (k, k)-choosable. This implies that every planar graph G is defectively $(1, 1)$-choosable if $g(G) \geq 8$ and $(2, 2)$-choosable if $g(G) \geq 6$. Skrekovski [22] proved that planar graphs with girth at least 5 are defectively $(4, 4)$-choosable.

Glebov and Zambalaeva [15] proved that every planar graph with girth at least 6 is acyclically P_6-free 2-colourable while in [4] it was proved that such a graph admits a list 2-colouring where any monochromatic component is a path with at most 15 vertices. In the case of planar graphs of girth 5, no results about P_k-free 2-colouring with fixed k are known. Borodin and Glebov [6] proved that every planar graph of girth 5 admits a 2-colouring where vertices of colour 1 form an independent set while vertices of colour 2 induce a forest (without any restrictions on the length of monochromatic paths). This result was slightly improved by Kawarabayashi and Thomassen [18] in terms of colouring extensions. Glebov and Zambalaeva [14] proved that every planar graph of girth 5 is τ-partitionable. A graph G is called *τ-partitionable* if for any positive integers a and b such that $a + b$ is the number of vertices in the longest path of G, there exists a 2-colouring of G such that any monochromatic path of colour 1 contains at most a vertices while any monochromatic path of colour 2 contains at most b vertices.

Quite naturally, Mihok [20] and the authors of [4] asked about the minimum integer $g^* > 3$ such that every planar graph of girth g^* admits a P_k-free 2-colouring for some constant integer k. By the results in [4,15], it follows that $5 \leq g^* \leq 6$.

In this paper we prove that $g^* = 5$. More precisely, our main result can be formulated as follows:

Theorem 1. *Every planar graph of girth at least 5 admits an acyclic P_8-free 2-colouring.*

Moreover, we present the list version of Theorem 1.

Theorem 2. *For any planar graph $G = (V, E)$ of girth at least 5 and for any list assignment L such that $|L(v)| = 2$ for every vertex $v \in V$ there exists an acyclic P_8-free L-colouring of G.*

Clearly, Theorem 2 implies Theorem 1 if $L(v) = \{1, 2\}$ for every $v \in V$. However, for the sake of presentation, we will prove Theorem 1 first and then modify its proof in order to establish Theorem 2 (which is technically a bit more complicated but can be derived using the same ideas).

Unlike most results mentioned above, our proof of Theorems 1 and 2 is not based on Euler's Formula, but is motivated by the proof of the well-known theorem that planar graphs are 5-choosable by Thomassen [23] and by the powerful technique of safe subgraphs developed by Borodin. The approach of Borodin is that a reducible configuration in a plane graph is found inside a subgraph bounded by a minimal separated cycle of a suitable length (see [1,5]). However, our proof method involves some new features compared to the approaches of Thomassen and Borodin. The detailed description of our technique along with the formulation of the main technical Lemma 1 is given in Section 2 of the paper. Section 3 represents the proof of Lemma 1. Section 4 is devoted to the proof of Theorem 2.

2. Specification of vertices and the main technical result

Suppose $G = (V, E)$ is a plane graph of girth at least 5 with the vertex set V and the edge set E. By F we denote the outer face of G and by $V(F)$ we denote the set of all vertices of G incident with F. We refer to the vertices in $V(F)$ as *external* vertices of G while the vertices in $V \setminus V(F)$ are *internal*.

As it was mentioned above, our proof of Theorems 1 and 2 follows the lines of the proof of Thomassen's theorem that every planar graph is 5-choosable. The main idea of Thomassen's proof is to put forward a stronger statement by imposing additional requirements on the colouring of vertices of the outer face of a plane graph. More specifically, such external vertices are assigned smaller lists of colours (of size 3 or 1) compared to the internal vertices, which are given lists of size 5.