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On excluded minors for classes of graphical matroids

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A R T I C L E I N F O

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A B S T R A C T

Frame matroids and lifted-graphic matroids are two distinct minor-closed classes of matroids, each of which generalises the class of graphic matroids. The class of quasi-graphic matroids, recently introduced by Geelen, Gerards, and Whittle, simultaneously generalises both the classes of frame and lifted-graphic matroids. Let $\mathcal{M}$ be one of these three classes, and let r be a positive integer. We show that $\mathcal{M}$ has only a finite number of excluded minors of rank r.

A matroid is a *frame matroid* if it may be extended so that it has a basis B such that every element is spanned by at most two elements of B. Such a basis is a *frame* for the matroid. A matroid M is a *lifted-graphic matroid* if there is a matroid N with $E(N) = E(M) \cup \{e\}$ such that $N \setminus e = M$ and N/e is graphic. These fundamental classes of matroids were introduced by Zaslavsky in a foundational series of papers [9–12], the former under the name of "bias matroids" as a generalisation of Dowling geometries. In [11] Zaslavsky defined the class of frame matroids as above and showed that the classes of bias and frame matroids are in fact the same.

Frame matroids are a natural generalisation of graphic matroids: the cycle matroid $M(G)$ of a graph G is naturally extended by adding its vertex set $V(G)$ as its frame, and declaring each non-loop edge to be minimally spanned by its endpoints. Classes of representable frame matroids play an important role in the matroid-minors project of Geelen, Gerards, and Whittle [6, Theorem 3.1], analogous to that of graphs embedded on surfaces in graph structure theory.

Frame matroids form a minor-closed class. Despite its importance, little is known about its excluded minors. Zaslavsky exhibited several in [11]. Bicircular matroids are a relatively well-studied proper minor-closed class of frame matroids; it is known that this class has only a finite number of excluded minors [5]. There are also natural proper minor-closed classes of frame matroids, and of lifted-graphic matroids, that have, for any fixed $r \geq 3$, infinitely many excluded minors of rank r [3]. The first systematic study of excluded minors for the class of frame matroids is [4], in which 18 excluded minors of connectivity 2 for the class of frame matroids is exhibited, and it is proved that any other excluded minor of connectivity 2 is a 2-sum of a 3-connected non-binary frame matroid with $U_{2,4}$. The class of lifted-graphic matroids is minor-closed. Less is known of their excluded minors than of those for frame matroids.

Here we prove the following theorems.

Theorem 1. *Let r be a positive integer. There are only a finite number of excluded minors of rank r for the class of frame matroids.*

Theorem 2. *Let r be a positive integer. There are only a finite number of excluded minors of rank r for the class of lifted-graphic matroids.*

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The prevailing belief among members of the matroid community was that each of these classes should have only a finite number of excluded minors. However, Chen and Geelen [2] have recently settled the question, rather surprisingly, by exhibiting, for each class, an infinite family of excluded minors. Each family consists of a sequence of matroids $(M_r)_{r\geq 7}$, defined using a sequence of graphs $(G_r)_{r\geq 7}$, with each excluded minor M_r having rank r. Theorems 1 and 2 say that, like Chen and Geelen's families, every infinite collection of excluded minors for these classes must contain matroids of arbitrarily large rank. Chen and Geelen point out that, "The existence of an infinite set of excluded minors does not necessarily prevent us from describing a class explicitly"; they point to Bonin's excluded minor characterisation of lattice-path matroids [1] as an encouraging example. Chen and Geelen's two infinite families of excluded minors are of a similar flavour to Bonin's infinite collections of excluded minors: Bonin's characterisation has three easily described infinite sequences of excluded minors, each consisting of a set of matroids indexed by the positive integers, of ever-increasing and unbounded ranks. Theorems 1 and 2, therefore, may be seen as support for Chen and Geelen's optimism.

In [7], Geelen, Gerards, and Whittle introduce the class of quasi-graphic matroids, as a common generalisation of the classes of frame and lifted-graphic matroids. For a vertex v, denote by $\text{loops}(v)$ the set of loops incident to v. Given a matroid M, a *framework* for M is a graph G satisfying

(1) $E(G) = E(M)$,
(2) for each component H of G, $r(E(H)) \leq |V(H)|$,
(3) for each vertex $v \in V(G)$, $\text{cl}(E(G - v)) \subseteq E(G - v) \cup \text{loops}(v)$, and
(4) if C is a circuit of M, then the graph induced by $E(C)$ has at most two components.

A matroid is *quasi-graphic* if it has a framework. Chen and Geelen conjecture that the class of quasi-graphic matroids has only finitely many excluded minors [2]. We prove that, like the classes of frame and of lifted-graphic matroids, when fixing the rank this is indeed the case.

Theorem 3. *Let r be a positive integer. There are only a finite number of excluded minors of rank r for the class of quasi-graphic matroids.*

1. Preliminaries

1.1. Frame matroids

Zaslavsky [11] has shown that the class of frame matroids is precisely that of matroids arising from *biased graphs*, as follows. Let M be a frame matroid on ground set E, with frame B. By adding elements in parallel if necessary, we may assume $B \cap E = \emptyset$. Hence for some matroid N, $M = N \setminus B$ where B is a basis for N and every element $e \in E$ is minimally spanned by either a single element or a pair of elements in B. Let G be the graph with vertex set B and edge set E, in which e is a loop with endpoint f if e is parallel with $f \in B$, and otherwise e is an edge with endpoints $f, f' \in B$ if $e \in \text{cl}\left(\{f, f'\}\right)$. The edge set of a cycle of G is either independent or a circuit in M. A cycle C in G whose edge set is a circuit of M is said to be *balanced*; otherwise C is *unbalanced*. Let $\mathcal{B}$ be the collection of balanced cycles of G. The pair $(G, \mathcal{B})$ is called a *biased graph*; one may think of the pair as a graph equipped with the extra information of the *bias* –balanced or unbalanced – of each of its cycles. A *theta graph* consists of a pair of distinct vertices and three internally disjoint paths between them. The circuits of M are precisely those sets of edges inducing one of: a balanced cycle, a theta subgraph in which all three cycles are unbalanced, two edge-disjoint unbalanced cycles intersecting in just one vertex, or two vertex-disjoint unbalanced cycles along with a minimal path connecting them. The later two biased subgraphs are called *handcuffs*, *tight* or *loose*, respectively. Such a biased graph $(G, \mathcal{B})$ *represents* the frame matroid M, and we write $M = F(G, \mathcal{B})$. Since the collection $\mathcal{B}$ of balanced cycles of G is determined by the matroid M, we may speak simply of the graph G as a frame representation of M, with its collection of balanced cycles being understood as implicitly given by M. Thus we may unambiguously refer to G as a *frame graph for M*, or say that M *has* a frame graph G. When it is clear from context that G is a frame representation of M, we say simply that G is a *graph* for M or that M *has* a graph G.

1.2. Lifted-graphic matroids

Let N be a matroid on ground set $E \cup \{e\}$, and suppose G is a graph with edge set E and with cycle matroid $M(G)$ equal to N/e. Then $M = N \setminus e$ is a lifted-graphic matroid. Each cycle in G is either a circuit of N, or together with e forms a circuit of N. Again, cycles whose edge set is a circuit of M are said to be balanced, and those whose edges form an independent set are unbalanced. Zaslavsky has shown [10] that the circuits of M are precisely those sets of edges inducing one of: a balanced cycle, a theta subgraph in which all three cycles are unbalanced, two edge disjoint unbalanced cycles meeting in just one vertex, or two vertex-disjoint unbalanced cycles. Letting $\mathcal{B}$ denote the collection of balanced cycles of G, we again say the biased graph $(G, \mathcal{B})$ so obtained *represents* the lifted-graphic matroid M; we write $M = L(G, \mathcal{B})$. As with frame matroids, we may more simply say G is a *lift graph for M*, or that M *has* a lift graph G, with its collection of balanced cycles being implicitly given by M. Similarly, when clear in context that G is a lift graph for M, we may say simply that G is a *graph* for M, and that M *has* the graph G.