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The graph grabbing game on $K_{m,n}$-trees

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ABSTRACT

The *graph grabbing game* is a two-player game on weighted connected graphs where all weights are non-negative. Two players, Alice and Bob, alternately remove a non-cut vertex from the graph (i.e., the resulting graph is still connected) and get the weight assigned to the vertex, where the starting player is Alice. Each player's aim is to maximize his/her outcome when all vertices have been taken, and Alice wins the game if she gathered at least half of the total weight. Seacrest and Seacrest (2017) proved that Alice has a winning strategy for every weighted tree with even order, and conjectured that the same statement holds for every weighted connected bipartite graph with even order. In this paper, we prove that Alice wins the game on a type of a connected bipartite graph with even order called a $K_{m,n}$*-tree*.

1. Introduction

In this paper, we only deal with finite simple undirected graphs. For a graph G and a subset $S \subseteq V(G)$, $G - S$ denotes the subgraph induced by the vertices in $V(G) \setminus S$. In particular, if S consists of a single vertex v, then we denote the induced subgraph by $G - v$. If $G - v$ is not connected, then v is called a *cut vertex*. A graph is *even* (resp. *odd*) if the number of vertices is even (resp. odd). A *weighted* graph G is a graph with a function $\omega : V(G) \to \mathbb{R}^+ \cup \{0\}$. In what follows, since we only deal with weighted graphs, i.e., each graph is already weighted, we omit "weighted" for simplicity.

In this paper, we consider a game on weighted graphs, called the *graph grabbing game* (or simply the *game*).

Graph grabbing game.

There are two players: Alice and Bob. Starting with Alice, they take the vertices alternately one by one and collect their weights. The vertices taken are removed from the graph. The choice of a vertex to be played in each move is restricted by the rule that after each move the remaining vertices form a connected graph (that is, each player cannot take a cut vertex). Both players' aim is maximizing their outcomes at the end of the game, when all vertices have been taken. Alice wins the game if she gets at least half of the total weight of the graph, and otherwise, Bob wins.

In the literature, the graph grabbing game is introduced in Winkler's puzzle book [9]. He showed that Alice can guarantee herself at least half of the weight of any even path. Moreover, he observed that there exists an odd path that Alice cannot obtain at least half of the total weight of it; for example, consider an odd path with three vertices in which the middle vertex has a positive weight and any other has weight zero. On the other hand, the graph grabbing game on cycles is called the *Pizza*

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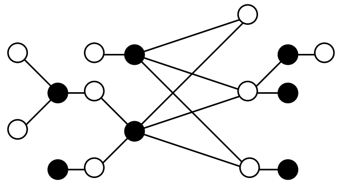

Fig. 1. A $K_{2,3}$-tree.

game. Similarly to the result by Winkler, Alice can always obtain at least half of the total weight of an even cycle. However, there exists an odd cycle such that Alice can obtain at most 4/9 of the total weight, and Alice can obtain at least 4/9 of the total weight of every odd cycle [2,5].

In general, for any given graph G, to decide which player wins the game on G is PSPACE-complete [3]. So, by restricting graph classes, we would like to know whether Alice can win the game on graphs in a given graph class. Micek and Walczak [7] proved that Alice can obtain at least $\frac{1}{4}$ of the total weight of every even tree, and conjectured that Alice can win the game on every even tree. Seacrest and Seacrest [8] verified the conjecture by an elegant proof, and they conjectured the following.

Conjecture 1 (*[8]*). *Alice can win the game on every weighted connected bipartite even graph.*

As far as we know, the conjecture is proved only for a few graph classes; cycles [2], trees [8] and complete bipartite graphs (easy). On the other hand, there are many graphs in which Alice cannot win the game on non-bipartite graphs and bipartite odd graphs [3]. For other variants of the graph grabbing game, see [1,4,6].

In this paper, we give a partial solution for Conjecture 1 focusing on a special graph, called a $K_{m,n}$-tree. A $K_{m,n}$-*tree* G with $m, n \geq 1$ is a bipartite graph obtained from a complete bipartite graph $K_{m,n}$ and trees $T_1, T_2, \ldots$ by identifying exactly a vertex of $K_{m,n}$ and exactly one of T_i for each i (see Fig. 1 for example).

Observe that a $K_{m,n}$-tree G is just a tree if $m = 1$ or $n = 1$. Furthermore, we can easily construct infinitely many odd $K_{m,n}$-trees in which Alice cannot win the game. The following is our main result.

Theorem 2. *Alice can win the game on every weighted even $K_{m,n}$-tree.*

The paper is organized as follows. In the next section, we prove Theorem 2 by an extended version of the proof technique used in [8]. In Section 3, we show that the proof technique in the proof of Theorem 2 cannot be directly applied to general even bipartite graphs.

2. Proof of Theorem 2

Inspired by the proof in [8], we consider a *rooted game* on a graph. In the rooted game, we are given a graph G and a *root set* $S \subseteq V(G)$. For a given root set $S \subseteq V(G)$, the *rooted game* on G is a graph grabbing game with the following rule:

During the game, any connected component contains at least one vertex in S.

In particular, if some connected component H contains a unique vertex v in S, then v must be grabbed at the last move of the game on H. (Note that the rooted game admits the resulting graph G' to be disconnected if each component in G' contains a vertex in S.)

The rooted game is introduced in [8] to study the game on a sub-tree of a given tree as a substructure of the original one. For example, we suppose that two sub-trees T' and T'' are obtained from T by removing an edge $uv \in E(T)$ in which $u \in V(T')$ and $v \in V(T'')$. In this case, the game on T can be considered as the combination of two rooted games on T' with root u and T'' with root v (until a player grabs one of u and v). So, we also analyze the rooted game to prove theorems for the game on $K_{m,n}$-trees. (In our definition of the root set S, each player may grab a vertex in S.)

We introduce some definitions. For a graph G, if a player takes $v \in V(G)$ at some move, then we say that the move is v. A move u in the game (resp. rooted game) on G (resp. $G_{R(S)}$) is *feasible* if $G - u$ is connected (resp. each component in $G - u$ has a root vertex), where S is the root set of G and $G_{R(S)}$ is the rooted graph with root set S. We denote by $\omega(v)$ the weight of v, and define the optimality of a move, as follows:

$$\mathrm{val}(G) = \begin{cases} 0 & \text{if } V(G) = \emptyset, \\ \max\{\omega(v) - \mathrm{val}(G - v) \mid v \text{ is feasible}\} & \text{otherwise.} \end{cases}$$

$$\text{A move } v \text{ is } \textit{optimal} \iff \mathrm{val}(G) = \omega(v) - \mathrm{val}(G - v)$$

In what follows, we may assume that both players *optimally* play the game, i.e., their moves are always optimal (we also say that a player plays by an optimal strategy).

The first (resp. second) player in the game on a graph G is denoted by 1 (resp. 2), and the player who moves at the last (resp. the second from the last) is denoted by -1 (resp. -2). Observe that if G is even, then $i = -j$, and otherwise, $i = -i$,