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Problems on matchings and independent sets of a graph

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A B S T R A C T

Let G be a finite simple graph. For $X \subset V(G)$, the *difference* of X, $d(X) := |X| - |N(X)|$ where $N(X)$ is the neighborhood of X and $\max\{d(X) : X \subset V(G)\}$ is called the *critical difference* of G. X is called a *critical set* if $d(X)$ equals the critical difference and $\ker(G)$ is the intersection of all critical sets. $\text{diadem}(G)$ is the union of all critical independent sets. An independent set S is an *inclusion minimal set with* $d(S) > 0$ if no proper subset of S has positive difference.

A graph G is called a *König–Egerváry* graph if the sum of its independence number $\alpha(G)$ and matching number $\mu(G)$ equals $|V(G)|$.

In this paper, we prove a conjecture which states that for any graph the number of inclusion minimal independent set S with $d(S) > 0$ is at least the critical difference of the graph.

We also give a new short proof of the inequality $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$.

A characterization of unicyclic non-König–Egerváry graphs is also presented and a conjecture which states that for such a graph G, the critical difference equals $\alpha(G) - \mu(G)$, is proved.

We also make an observation about $\ker(G)$ using Edmonds–Gallai Structure Theorem as a concluding remark.

1. Introduction

In this paper G is a finite simple graph with the vertex set $V(G)$ and the edge set $E(G) \subset \binom{V(G)}{2}$. For $A \subset V(G)$, neighborhood of A, denoted by $N_G(A)$, is the set of all vertices adjacent to some vertex in A. When there is no confusion the subscript G may be dropped. The *degree* of a vertex v is the number of edges incident to v and is denoted by $\deg(v)$. If $U \subset V(G)$, then $G[U]$ is the graph with the vertex set U and whose edges are precisely the edges of G with both ends in U. A set S of vertices is *independent* if no two vertices from S are adjacent. The cardinality of a largest independent set (*maximum independent set*) is denoted by $\alpha(G)$. The reader may refer to any of the standard text books [2,11] for basic notations.

This paper is based on the results in [3–10,12,14]. We state the following definitions which will help us to formulate the results proved in this paper.

Let $\text{ind}(G) = \{S : S \text{ is an independent set of } G\}$, $\Omega(G) = \{S \in \text{ind}(G) : |S| = \alpha(G)\}$ and $\text{core}(G) = \cap\{S : S \in \Omega(G)\}$[5].

For $X \subset V(G)$, the number $|X| - |N(X)|$ is called the *difference* of the set X and is denoted by $d_G(X)$. Again we drop the subscript G in case of no ambiguity. The number $\max\{d(X) : X \subset V(G)\}$ is called the *critical difference* of G and is denoted by $d_c(G)$. A set $U \subset V(G)$ is *critical* if $d(U) = d_c(G)$ [14] and $\ker(G)$ is the intersection of all critical sets [7]. *Diadem* of a graph is defined as the union of all critical independent sets and it is denoted by $\text{diadem}(G)$ [3]. One may observe that

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$\ker(G) \subset \text{core}(G)$ [7] and thus $\ker(G) \in \text{ind}(G)$. An independent set S is an inclusion minimal set with $d(S) > 0$ if no proper subset of S has positive difference [9].

A matching in a graph G is a set M of edges such that no two edges in M share a common vertex. Size of a largest possible matching (*maximum matching*) is denoted by $\mu(G)$. A vertex is *matched* (or *saturated*) by M if it is an endpoint of one of the edges in M. A perfect matching is a matching which matches all vertices of the graph. For two disjoint subsets A and B of $V(G)$ we say that there is a *matching from A into B* if there is a matching M such that any edge in M joins a vertex in A and a vertex in B and all the vertices in A are matched by M.

A graph G is a König–Egerváry (KE) graph if $\alpha(G) + \mu(G) = |V(G)|$ [1,13]. König–Egerváry graphs have been well studied. Levit and Mandrescu studied the critical difference, ker, core, diadem of graphs, properties of König–Egerváry graphs and proved several results. Based on these results several natural conjectures and problems arose. The ones considered in this paper are stated below.

Conjecture 1.1 (*[9]*)**.** *For any graph G, the number of inclusion minimal independent set S such that* $d(S) > 0$ *is at least* $d_c(G)$*.*

Theorem 1.2. *For any graph G,* $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$*.*[1]

Conjecture 1.3. *For a unicyclic non-KE graph G,* $d_c(G) = \alpha(G) - \mu(G)$*.*[2]

Problem 1.4 (*[3,8]*)**.** Characterize graphs such that $\text{core}(G)$ is critical.

Problem 1.5 (*[3,8]*)**.** Characterize graphs with $\ker(G) = \text{core}(G)$.

In Section 2, Conjecture 1.1 and related results are proved. In Section 3 a new short proof of Theorem 1.2 is given. A characterization of unicyclic non-König–Egerváry graph is presented in Section 4 and as a corollary Conjecture 1.3 is deduced. In the concluding section Edmonds–Gallai structure Theorem is used to make an observation regarding $\ker(G)$. It may be useful for Problems 1.4 and 1.5.

2. On minimum number of inclusion minimal sets with positive difference

In this section we study $X \in \text{ind}(G)$ with $d(X) > 0$ such that for all $Y \subsetneq X$, $d(Y) < d(X)$ and give a proof of Conjecture 1.1. There are several results that led to the formulation of this conjecture. Some of them are listed below as they help to understand the proof or they are used in the proof of the conjecture.

Theorem 2.1 (*[4]*)**.** *There is a matching from* $N(S)$ *into* S *for every critical independent set* S*.*

Theorem 2.2 (*[7]*)**.** *For every graph G, the following assertions are true:*

(i) $\ker(G)$ *is the unique minimal critical independent set of G.*
(ii) $\ker(G) \subset \text{core}(G)$.

Theorem 2.3 (*[7]*)**.** *For a graph G, the following assertions are true:*

(i) *The function d is supermodular, i.e.,* $d(X \cup Y) + d(X \cap Y) \geq d(X) + d(Y)$ *for every* $X, Y \subset V(G)$*.*
(ii) *If X and Y are critical in G, then* $X \cup Y$ *and* $X \cap Y$ *are critical as well.*

Theorem 2.4 (*[8]*)**.** *If G is a bipartite graph, then* $\ker(G) = \text{core}(G)$*.*

Theorem 2.5 (*[9]*)**.** *For a vertex v in a graph G, the following assertions hold:*

(i) $d_c(G - v) = d_c(G) - 1$ *if and only if* $v \in \ker(G)$ *;*
(ii) *if* $v \in \ker(G)$*, then* $\ker(G - v) \subset \ker(G) - v$*.*

Theorem 2.6 (*[9]*)**.** *If* $\ker(G) \neq \emptyset$*, then* $\ker(G) = \cup\{S : S$ *is an inclusion minimal independent set with* $d(S) > 0\}$*.*

For an independent set X of G a new graph H_X is defined as follows. The vertex set $V(H_X) = X \cup N(X) \cup \{v, w\}$, where v and w are two new vertices not in $V(G)$ and the edge set $E(H_X) = \{xy \in E(G) : x \in X, y \in N(X)\} \cup \{vw\} \cup \{vx : x \in N(X)\}$. Note that if G is a connected graph with $|V(G)| > 1$, then H_X is a connected bipartite graph. Fig. 1 gives an illustration of the construction. Also observe that for all $Y \subset X$, $d_{H_X}(Y) = d_G(Y)$.

Theorem 2.7. *If X is an independent set of G with* $d(X) > 0$ *such that for all* $Y \subsetneq X$*,* $d(Y) < d(X)$*, then* $\ker(H_X) = X$*.*

[1] Conjectured in [3] and proved in [12].
[2] Stated by Levit in a talk at Tata Institute of Fundamental Research in 2014.