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Compatible Eulerian circuits in Eulerian (di)graphs with generalized transition systems[☆]

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ABSTRACT

A transition in a graph is defined as a pair of adjacent edges. A transition system of an Eulerian graph refers to a set of partitions such that for each vertex of the graph, there corresponds to a partition of the set of edges incident to the vertex into transitions. A generalized transition system $F(G)$ over a graph G defines a set of transitions over G. A compatible Eulerian circuit of an Eulerian graph G with a generalized transition system $F(G)$ is defined as an Eulerian circuit in which no two consecutive edges form a transition defined by $F(G)$. In this paper, we further introduce the concept of weakly generalized transition system which is an extension of the generalized transition system and prove some Ore-type sufficient conditions for the existence of compatible Eulerian circuits in Eulerian graphs with (weakly) generalized transition systems and obtain corresponding results for Eulerian digraphs. Our conditions improve some previous results due to Jackson and Isaak, respectively.

1. Introduction

In this paper we consider only finite graphs (digraphs) without loops, but possibly containing multiple edges (arcs). For terminology and notations not defined here, we refer the reader to Bondy and Murty [2].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and edges of G, respectively. For a vertex $v \in V(G)$, denote by $E_G(v)$ the set of edges of G incident to v and by $d_G(v) = |E_G(v)|$ the *degree* of v in G. If the graph G is understood, we will use the notation $E(v)$ instead of $E_G(v)$ and $d(v)$ instead of $d_G(v)$. The set of vertices of degree at least four in G is denoted by $V_4(G)$. The *line graph* of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if they are adjacent in G. An *Eulerian circuit* refers to a closed trail that traverses each edge of a graph. A graph is said to be *Eulerian* if it admits an Eulerian circuit. Two edges are said to be *consecutive* with respect to a trail if they are traversed consecutively along the trail.

For a vertex v of a graph G, denote by $P(v)$ a partition of the set $E(v)$. If a partition $P(v)$ exists for each vertex v of G, we will say that G has a *partition system*, denoted by $P(G) = \{P(v) : v \in V(G)\}$. An Eulerian circuit X of an Eulerian graph G is said to be *compatible* to a partition system $P(G)$ if and only if each pair of consecutive edges of X through v are in distinct classes of $P(v)$ for each $v \in V(G)$. In an Eulerian graph G with a partition system $P(G)$, a *compatible Eulerian circuit* of G refers to an Eulerian circuit compatible to $P(G)$. Kotzig [15] first gave a necessary and sufficient condition for the existence of compatible Eulerian circuits in Eulerian graphs with partition systems, as follows.

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Theorem 1 (*Kotzig [15]*). *Let G be an Eulerian graph with a partition system $P(G)$. Then, a compatible Eulerian circuit exists if and only if the size of every class of $P(v)$ is at most $d(v)/2$ for each $v \in V(G)$.*

An *edge-coloring* of a graph G is defined as a mapping $C : E(G) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. An *edge-colored graph* refers to a graph with a fixed edge-coloring. An edge-coloring of G may define, in an obvious way, a partition system $P(G)$ in which, for every vertex $v \in V(G)$, each class of $P(v)$ is the set of edges of $E(v)$ with a same color. Thus, a compatible Eulerian circuit in an edge-colored Eulerian graph is an Eulerian circuit in which each pair of consecutive edges have distinct colors. Denote by $C(G)$ the set of colors appearing on the edges of an edge-colored graph G. We use $d^i(v)$ to denote the size of the set $\{e \in E(v) : C(e) = i\}$ for a vertex $v \in V(G)$ and a color $i \in C(G)$. Benkouar et al. [1] proved that there exists a compatible Eulerian circuit in an edge-colored Eulerian graph G if and only if $d^i(v) \leq \sum_{j\neq i} d^j(v)$ holds for each vertex $v \in V(G)$ and each color $i \in C(G)$, and provided a polynomial time algorithm to find it.

Jackson [12] introduced the concept of transition system, as a special case of a partition system for a graph. For a vertex v of a graph G with a partition system $P(G)$, a class of $P(v)$ with two edges is called a *transition*, and a *transition decomposition* $T(v)$ at v is defined as a partition of $E(v)$ into transitions. A *transition system* of an Eulerian graph refers to a partition system of the graph in which each class is a transition. Denote by $T(G) = \{T(v) : v \in V(G)\}$ a transition system of an Eulerian graph G. Two transition systems $T_1(G) = \{T_1(v) : v \in V(G)\}$ and $T_2(G) = \{T_2(v) : v \in V(G)\}$ are said to be *compatible* if $T_1(v) \cap T_2(v) = \emptyset$ for each $v \in V(G)$. There is a natural bijection between transition systems of an Eulerian graph and closed trail decompositions of the graph (a partition of the edge set into closed trails): two edges form a transition if and only if they are consecutive in a closed trail. We use $T(\mathcal{C})$ to denote the transition system associated with the closed trail decomposition $\mathcal{C}$. In particular, each Eulerian circuit X defines a transition system $T(X)$ associated with X. Thus, the compatibility of two Eulerian circuits can be similarly defined. Jackson [12] considered the problem of deciding when a given Eulerian graph contains several pairwise compatible Eulerian circuits (see also [6,8,9,14]).

Jackson [13] studied a variation of the above problem by giving a set of transitions over an Eulerian graph, and found sufficient conditions for the existence of an Eulerian circuit in which no two consecutive edges form a transition in the given set. A *generalized transition system* over a graph G is defined as a set of mappings $F(G) = \{F_v : v \in V(G)\}$ such that for each $v \in V(G)$, $F_v : E(v) \to 2^{E(v)}$ and whenever $h \in F_v(g)$ we have $g \in F_v(h)$. An Eulerian circuit of an Eulerian graph G is said to be *compatible* to a generalized transition system $F(G)$ if no two consecutive edges in the Eulerian circuit form a transition defined by $F(G)$. In an Eulerian graph G with a generalized transition system $F(G)$, a *compatible Eulerian circuit* of G refers to an Eulerian circuit compatible to $F(G)$. Denote by $f_v(g)$ the size of $F_v(g)$ for $g \in E(v)$ and $v \in V(G)$. Jackson [13] provided a Dirac-type sufficient condition for the existence of compatible Eulerian circuits in Eulerian graphs with generalized transition systems, as follows.

Theorem 2 (*Jackson [13]*). *Let G be an Eulerian graph with a generalized transition system $F(G)$. For each incident pair (v, g), if $f_v(g) \leq d(v)/2 - 1$ for $d(v) \equiv 0 \pmod 4$ or $d(v) = 2$, and $f_v(g) \leq d(v)/2 - 2$ otherwise, then there exists a compatible Eulerian circuit in G.*

Dvořák [5] proved that finding a compatible Eulerian circuit in an Eulerian graph (and Eulerian digraph) with a generalized transition system is NP-complete. Recently, Krivelevich, Lee and Sudakov [16] presented a different description for the concept of generalized transition system. A generalized transition system $F(G)$ is said to be *k-bounded* if $f_v(g) \leq k$ for every $g \in E(v)$ and $v \in V(G)$. The authors [16] pointed out that a k-bounded generalized transition system over a graph can be viewed as a quantitative measure of robustness of graph properties and further study of how various extremal results can be strengthened using this concept appears to be a promising direction of research.

More naturally, we can introduce the concept of weakly generalized transition system as follows. A *weakly generalized transition system* over a graph G is defined as a set of mappings $\vec{F}(G) = \{F_v^+ : v \in V(G)\}$ such that for each $v \in V(G)$, $F_v^+ : E(v) \to 2^{E(v)}$ and whenever $h \in F_v^+(g)$ it is not necessary that $g \in F_v^+(h)$. An oriented Eulerian circuit of an Eulerian graph G is said to be *compatible* to a weakly generalized transition system $\vec{F}(G)$ if no two consecutive edges in the oriented Eulerian circuit form an oriented transition defined by $\vec{F}(G)$. In an Eulerian graph G with a weakly generalized transition system $\vec{F}(G)$, a *compatible oriented Eulerian circuit* of G refers to an oriented Eulerian circuit compatible to $\vec{F}(G)$. Moreover, for $g \in E(v)$ and $v \in V(G)$, we define $F_v^-(g) = \{h : g \in F_v^+(h)\}$, and use $f_v^+(g)$ and $f_v^-(g)$ to denote the size of $F_v^+(g)$ and $F_v^-(g)$, respectively.

It is not difficult to see that for each edge-coloring of a graph G, there exists a trivial partition system $P(G)$ such that each class of $P(v)$ consists of all edges of $E(v)$ with a same color. But the converse is not true. For some partition system of a graph, there exists no such edge-coloring of the graph corresponding to it. For example, look at the graph G in Fig. 1(a). If the partition system is $P(G) = \{P(v_1), P(v_2), P(v_3)\}$, where $P(v_1) = \{\{e_1, e_3\}\}$, $P(v_2) = \{\{e_1, e_2\}\}$ and $P(v_3) = \{\{e_2\}, \{e_3\}\}$, then we cannot find an edge-coloring corresponding to it. It is trivial that each partition system defines a generalized transition system. Also, the converse is not true. For example, look at the graph G in Fig. 1(b). If the generalized transition system is $F(G) = \{F_u\} \cup \{F_{v_i} : i = 1, 2, 3\}$, where $F_u(e_1) = \{e_2, e_3\}$, $F_u(e_2) = \{e_1\}$ and $F_u(e_3) = \{e_1\}$, and $F_{v_1}(e_1) = F_{v_2}(e_2) = F_{v_3}(e_3) = \emptyset$, then we cannot find a partition system corresponding to it. Moreover, replacing each transition in a generalized transition system with two oriented transitions with different orientations, we can get a weakly generalized transition system. Thus a weakly generalized transition system is a more general form.

For the sake of brevity, we just define a digraph to be a graph where the orientations of edges have been added, translating all the concepts defined above for graphs to the case of digraphs. Let D be a digraph. Denote by $V(D)$ and $A(D)$ the sets of