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The expansion of a chord diagram and the Tutte polynomial

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ABSTRACT

A chord diagram is a set of chords of a circle such that no pair of chords has a common endvertex. A chord diagram E is called nonintersecting if E contains no crossing. For a chord diagram E having a crossing $S = \{x_1x_3, x_2x_4\}$, the expansion of E with respect to S is to replace E with $E_1 = (E \setminus S) \cup \{x_2x_3, x_4x_1\}$ or $E_2 = (E \setminus S) \cup \{x_1x_2, x_3x_4\}$. For a chord diagram E, let $f(E)$ be the chord expansion number of E, which is defined as the cardinality of the multiset of all nonintersecting chord diagrams generated from E with a finite sequence of expansions.

In this paper, it is shown that the chord expansion number $f(E)$ equals the value of the Tutte polynomial at the point $(2, -1)$ for the interlace graph G_E corresponding to E. The chord expansion number of a complete multipartite chord diagram is also studied. An extended abstract of the paper was published (Nakamigawa and Sakuma, 2017) [13].

1. Introduction

A *chord diagram* is a set of chords of a circle, in which no pair of chords shares a common endvertex. A *crossing* is a pair of chords crossing each other in a circle. Let V be a set of $2n$ vertices on a circle, and let E be a chord diagram of order n, where each chord has endvertices of V. We call V the support of E. We denote the family of all chord diagrams having V as their supports by $\mathcal{CD}(V)$.

Let $E \in \mathcal{CD}(V)$. Suppose that $x_1, x_2, x_3, x_4 \in V$ are placed on a circle in clockwise order. For a crossing $S = \{x_1x_3, x_2x_4\} \subset E$, let $S_1 = \{x_2x_3, x_4x_1\}$, and $S_2 = \{x_1x_2, x_3x_4\}$. The *expansion* of E with respect to S is defined as a replacement of E with $E_1 = (E \setminus S) \cup S_1$ or $E_2 = (E \setminus S) \cup S_2$.

If a chord diagram contains no crossing, it is called *nonintersecting*. For a chord diagram $E \in \mathcal{CD}(V)$ and a crossing $S \subset E$, we have two successors E_1 and E_2 of E. By iterating possible expansions, we have a multiset of nonintersecting chord diagrams. As is remarked in [12], the resulting multiset of nonintersecting chord diagrams generated from E is uniquely determined.

For $E \in \mathcal{CD}(V)$, let us denote the multiset of nonintersecting chord diagrams generated from E by $\mathcal{NCD}(E)$. Furthermore, let us define the *chord expansion number* $f(E)$ of E as the cardinality of $\mathcal{NCD}(E)$ as a multiset.

Let C_n be an n-crossing, which consists of n chords crossing each other. In [12], it is proved that

$$\sum_{n\geq 0} f(C_n)\frac{x^n}{n!} = (\sec x + \tan x)' = \frac{1}{1-\sin x}$$
$$= 1 + x + 2\frac{x^2}{2!} + 5\frac{x^3}{3!} + 16\frac{x^4}{4!} + 61\frac{x^5}{5!} + 272\frac{x^6}{6!} + 1385\frac{x^7}{7!} + \cdots . \quad (1)$$

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Hence, we have $f(C_n) = Z_{n+1}$, where Z_n is called the Euler number, also called the Euler zigzag number, which corresponds to the number of alternating permutations of order n (see [15] for an excellent survey of alternating permutations).

2. Main results

For a chord diagram E, the *interlace graph* G_E of E is defined as a graph such that (1) a chord e of E corresponds to a vertex e^* of G_E, and (2) for two vertices e^* and f^* of G_E, e^*f^* is an edge of G_E if and only if their corresponding chords e and f of E are crossing. By definition, if E is an n-crossing, G_E is a complete graph K_n with n vertices. An interlace graph is also called a *circle graph*, and it has been extensively studied. (See [3,6,14].)

For a graph G, the *Tutte polynomial* $t(G; x, y)$ of G is known as a fundamental counting function. For example, for a connected graph G, $t(G; 1, 1)$ equals the number of spanning trees of G, $t(G; 2, 1)$ equals the number of spanning forests of G, $t(G; 1, 2)$ counts the number of connected subgraph of G, and $t(G; 2, 0)$ counts the number of acyclic orientations of G. (For the definition and basic properties of the Tutte polynomial, see, for example, [4,5,16].) Moreover, as one of unexpected results, it is known that $t(G; 3, 3)$ counts the number of T-tetromino tilings of a rectangular region [8].

Our main result is that the chord expansion number can be expressed by the value of the Tutte polynomial at the point $(2, -1)$.

Theorem 1. *For a chord diagram E, let G_E be the interlace graph corresponding to E. Then $f(E) = t(G_E; 2, -1)$.*

The proof will be given in Section 3.

The interlace polynomial, which was introduced in [1,2], is also a well-known counting function for graphs. ([10] is a nice survey of the interlace polynomial.) Hence, it is a natural question whether or not the chord expansion number is calculated by the interlace polynomial. As we will see later in the proof of Theorem 1, the expansion operation of a chord diagram is closely related to the deletion–contraction operation, by which the Tutte polynomial is defined, of the corresponding interlace graph. On the other hand, the expansion operation seems not directly associated to the pivot operation, also known as the switching operation, by which the interlace polynomial is defined. This may be one reason why, at least until now, we have not found a formula counting the chord expansion number by using the interlace polynomial.

By Theorem 1, in order to calculate the chord expansion number for a given chord diagram E, what we want to know is the value of the Tutte polynomial at the point $(2, -1)$. In [9], Merino proved that $t(K_n; 2, -1) = Z_{n+1}$, the Euler number, for $n \geq 1$. Hence, by Theorem 1, we have Eq. (1) in Section 1 as a corollary.

For a graph $G = (V, E)$ and $A \subset E$, let $G \mid_A = (V, A)$. Let $r(A)$ denote the rank of A, the number of edges of a spanning forest of $G \mid_A$, and let $n(A) = |A| - r(A)$, the nullity of A. It is known that the Tutte polynomial $t(G; x, y)$ is expressed as

$$t(G; x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{n(A)}$$

By this formula and Theorem 1, for a chord diagram C, we have

$$f(C) = \sum_{A \subset E(G_C)} (-2)^{n(A)}$$

In [7], Goodall, Merino, de Mier, and Noy deeply studied the value of the Tutte polynomial at the point $(2, -1)$. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. For a connected graph G with $V(G) = \{v_1, v_2, \ldots, v_s\}$, $\boldsymbol{c} = (c_1, c_2, \ldots, c_s) \in \{0, 1\}^s$ and $\boldsymbol{n} = (n_1, n_2, \ldots, n_s) \in \mathbb{Z}_+^s$, let us construct a new graph $G(\boldsymbol{c}; \boldsymbol{n})$ as follows:

(1) the vertex set of $G(\boldsymbol{c}, \boldsymbol{n})$ is $V_1 \cup V_2 \cup \cdots \cup V_s$, where $V_i \cap V_j = \emptyset$ for all $1 \leq i < j \leq s$ and $|V_i| = n_i$ for all $1 \leq i \leq s$, and

(2) for $u \in V_i$ and $v \in V_j$ with $1 \leq i \leq s$ and $1 \leq j \leq s$, uv is an edge of $G(\boldsymbol{c}; \boldsymbol{n})$ if and only if $i = j$ and $c_i = 1$, or $i \neq j$ and $v_iv_j \in E(G)$.

For a set of variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_s)$, let us define

$$T(x, y; \boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{Z}_+^s} t(G(\boldsymbol{c}; \boldsymbol{n}); x, y) \prod_{j=1}^{s} \frac{x_j^{n_j}}{n_j!}.$$

Goodall et al. proved the following result.

Theorem A *([7]). For a connected graph G of order s and for $\boldsymbol{c} \in \{0, 1\}^s$,*

$$T(2, -1; \boldsymbol{x}) = \left(\sum_{\boldsymbol{n} \in \mathbb{Z}_+^s} (-1)^{q(\boldsymbol{n})} \prod_{j=1}^{s} \frac{x_j^{n_j}}{(-2)^{n_j} n_j!} \right)^{-2},$$

where $q(\boldsymbol{n}) = \sum_{1 \leq i \leq s} \binom{n_i}{2} + \sum_{v_iv_j \in E(G)} n_in_j$, which is the number of edges of $G(\boldsymbol{c}; \boldsymbol{n})$.

Note that Theorem A is not explicitly appeared in [7], but is followed from Lemma 3.1 with $y = -1$ and Lemma 3.2 with $z = -2$, $w = -2$ of [7].

In order to apply Theorem A for chord diagrams, we note the following lemma.