Discrete Mathematics 341 (2018) 2113–2116

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Existence of a P_{2k+1}-decomposition in the Kneser graph $KG_{t,2}$

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A R T I C L E I N F O

Article history:
Received 22 June 2017
Received in revised form 16 April 2018
Accepted 17 April 2018

Keywords:
Kneser graph
Path decomposition

A B S T R A C T

In this paper, it is shown that if $t \equiv 0,\ 1,\ 2$ or $3 (\bmod 8k)$, $k \geq 2$, then the Kneser graph $KG_{t,\,2}$ can be decomposed into paths of length $2k$. Consequently, we obtain the following: for $k = 2^{\ell}$, $\ell \geq 1$, the Kneser graph $KG_{t,\,2}$ has a P_{2k+1}-decomposition if and only if $t \equiv 0,\ 1,\ 2$ or $3\ (\bmod\ 2^{\ell+3})$. Using this, the main result of the paper (Whitt III and Rodger, 2015) is deduced as a corollary.

1. Introduction

All graphs considered here are simple and finite. Let P_k (resp. C_k) denote the path (resp. cycle) on k vertices. For a graph $G = (V, E)$, and $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $\langle S\rangle$. Similarly, for $E' \subseteq E(G)$, the subgraph of G induced by E' is denoted by $\langle E'\rangle$. If $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G such that $E(G) = E(H_1)\cup E(H_2)\cup\cdots\cup E(H_k)$, then we say that $H_1, H_2, \ldots, H_k$ *decompose* G and we write this as $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$. If each $H_i \simeq H$, $1 \leq i \leq k$, then we say that H *decomposes* G and we denote it by $H \mid G$. If each $H_i \simeq C_k$ (resp. $H_i \simeq P_{k+1}$), the cycle (resp. path) of length k, then we write $C_k \mid G$ (resp. $P_{k+1} \mid G$) and in this case we say that G admits a C_k-*decomposition* (resp P_{k+1}-*decomposition*).

Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_0,\ x_1,\ \ldots, x_{r-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{r-1}\}$. If G contains the set of edges $F_i(X, Y) = \{x_j\, y_{j+i} \mid 0 \leq j \leq r-1\}$, $0 \leq i \leq r-1$, where addition in the subscript is taken modulo r, then we say that G has the *1-factor of jump i from X to Y*. Note that $F_i(X,\ Y) = F_{r-i}(Y,\ X)$. Clearly, if $G = K_{r,r}$, then $E(G) = \bigcup_{i=0}^{r-1} F_i(X, Y)$. Let $E(X,\ Y)$ denote the set of edges of G having one end in X and the other end in Y, where $X,\ Y \subset V(G)$ and $X \cap Y = \emptyset$.

Let $\mathcal{P}_k(t)$ be the set of all k-element subsets of a t element set. The *Kneser graph* $KG_{t,\,k}$ is defined as follows: $V(KG_{t,\,k}) = \mathcal{P}_k(t)$ and $E(KG_{t,\,k}) = \{AB \mid A, B \in \mathcal{P}_k(t)$ and $A \cap B = \emptyset\}$. Note that when $k = 1$, $KG_{t,\,k} \cong K_t$ and the graph $KG_{t,\,2}$ is isomorphic to $\overline{L(K_t)}$, where $L(K_t)$ denotes the line graph of K_t. Definitions which are not given here can be found in [1].

The Kneser graph was introduced by Kneser, see [5]. Initially, Kneser conjunctured that if $t \geq 2k$, then $\chi(KG_{t,\,k}) = t - 2k + 2$, where $\chi(KG_{t,\,k})$ denotes the chromatic number of $KG_{t,\,k}$. This conjuncture was settled using different proof techniques, see [2,4,5,7,8]. Another interesting problem is to prove that $KG_{t,\,k}(\neq K_{5,\,2})$ is Hamiltonian. Various authors considered this problem, see [3,10], but the problem is still open. In [9], Rodger and Whitt discussed necessary and sufficient conditions for the existence of a P_4-decomposition of $KG_{t,\,2}$. In the same paper, they considered P_4-decomposition of the generalized Kneser graph $GKG_{t,\,3,\,1}$, where the graph $GKG_{t,\,3,\,1}$ has $V(KG_{t,\,3,\,1}) = \mathcal{P}_3(t)$ and $E(K_{t,\,3,\,1}) = \{AB \mid A, B \in \mathcal{P}_3(t)$ with $|A \cap B| = 1\}$. Further, in [13], Whitt and Rodger proved that $P_5 \mid KG_{t,\,2}$ if and only if $t \equiv 0,\ 1,\ 2,\ 3 (\bmod 16)$.

We prove the following theorem.

Theorem 1.1. *If $t \geq 1$ is an integer such that $t \equiv a (\bmod 8k)$, $a \in \{0,\ 1,\ 2,\ 3\}$, $k \geq 2$, then $KG_{t,\,2}$ admits a P_{2k+1}-decomposition.*

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https://doi.org/10.1016/j.disc.2018.04.021

Here we quote some known results for our future reference.

Theorem 1.2 ([6]). *Let $n \geq 6$ be an even integer. Then the complete bipartite graph $K_{n,n} - E(F)$ is Hamilton cycle decomposable, where $E(F)$ is the edge set of any 2-factor F of $K_{n,n}$.* $\square$

Theorem 1.3 ([11]). *Let k, n be positive integers. Then $P_{k+1} \mid K_n$ if and only if $2 \leq k+1 \leq n$, and $n(n-1) \equiv 0 \pmod{2k}$.* $\square$

Theorem 1.4 ([12]). *Let m, n be positive even integers with $n \leq m$ and let $k \geq 2$ be an integer. Then $P_{k+1} \mid K_{m,n}$ if and only if $m \geq \lceil \frac{k+1}{2} \rceil$, $n \geq \lceil \frac{k}{2} \rceil$ and $k \mid mn$.* $\square$

2. Some useful lemmas

Lemma 2.1. *If $k \geq 2$ and $s \geq 1$, then the graph $KG_{8ks,2}$ admits a P_{2k+1}-decomposition.*

Proof. From the definition of $KG_{8ks,2}$, $V(KG_{8ks,2}) = \mathcal{P}_2(8ks)$. Let $V_0 = \{\{1, 2\}, \{3, 8ks\}, \{4, 8ks-1\}, \{5, 8ks-2\}, \dots, \{4ks+1, 4ks+2\}\}$ be a subset of $V(KG_{8ks,2})$. Consider the permutation $\rho = (1)(2\,3\,4 \dots 8ks)$ on the set $\{1, 2, \dots, 8ks\}$. Let $V_i = \rho^i(V_0)$, $1 \leq i \leq 8ks-2$, that is, $V_i = \{\{\rho^i(1), \rho^i(2)\}, \{\rho^i(3), \rho^i(8ks)\}, \dots, \{\rho^i(4ks+1), \rho^i(4ks+2)\}\}$. Clearly, $V_i \cap V_j = \emptyset$, $0 \leq i < j \leq 8ks-2$, $\mathcal{P}_2(8ks) = \bigcup_{i=0}^{8ks-2} V_i$ and $|V_i| = 4ks$. Since any pair of sets in V_i have empty intersection, $\langle V_i \rangle$ is a complete subgraph of $KG_{8ks,2}$ of order $4ks$ and $\langle V_i \rangle$ has a P_{2k+1}-decomposition, by Theorem 1.3. The set of edges of $KG_{8ks,2}$ which are not in $\bigcup_{i=0}^{8ks-2} \langle V_i \rangle$ is $\bigcup_{0 \leq i < j \leq 8k-2} E(V_i, V_j)$. Let $G(V_i, V_j) = \langle E(V_i, V_j) \rangle$, $0 \leq i < j \leq 8ks-2$; it is a bipartite subgraph of $KG_{8ks,2}$. Observe that, in $G(V_i, V_j)$, each vertex $\{a, b\}$ in V_i is adjacent to all vertices in V_j except the two vertices of the type $\{a, a'\}$, $a' \neq b$, and $\{b, b'\}$, $b' \neq a$. Consequently, $G(V_i, V_j)$ is a $(4ks-2)$-regular graph. As the graph $G(V_i, V_j)$ is isomorphic to $K_{4ks,4ks} - E(F)$, where F is a 2-factor of $K_{4ks,4ks}$, it admits a C_{8ks}-decomposition, by Theorem 1.2, and each C_{8ks} can be decomposed into $4s$ copies of P_{2k+1}. Thus $P_{2k+1} \mid KG_{8ks,2}$. $\square$

For our future reference we define two graphs.

(1) Let $k \geq 2$ and $s \geq 1$ and let $X = \{x_1, x_2, \dots, x_{4ks}\}$, $Y = \{y_1, y_2, \dots, y_{4ks}\}$ and $Z = \{z_1, z_2, \dots, z_{4ks}\}$. Let H_1 be the graph with $V(H_1) = X \cup Y \cup Z$ and $\langle X \cup Y \rangle \simeq \langle X \cup Z \rangle \simeq K_{4ks,4ks} - F_0$, where F_0 denotes the 1-factor of jump 0 in $K_{4ks,4ks}$.

(2) Let $k \geq 2$ and $s \geq 1$ and let $X = \{x_1, x_2, \dots, x_{4ks}\}$ and $Y = \{y_1, y_2, \dots, y_{4ks}\}$. Let H_2 be the graph with vertex set $\{\infty\} \cup X \cup Y$ and the adjacency is defined as follows: $\langle X \rangle$ and $\langle Y \rangle$ are independent sets in H_2 and the vertex ∞ is adjacent to all the vertices of X and, $\langle X \cup Y \rangle \simeq K_{4ks,4ks} - F_0$, where F_0 is the 1-factor of jump 0 in $K_{4ks,4ks}$.

Lemma 2.2. *For $k \geq 2$, the graph H_1 admits a P_{2k+1}-decomposition.*

Proof. Let H_1 be the graph described above. Consider the path $P = y_1 x_{4ks} y_2 x_{4ks-1} y_3 x_{4ks-2} \dots y_{k-1} x_{4ks-k+2} y_k x_{4ks-k+1} z_{4ks-k+2}$ of length $2k$ in H_1. Then $P, \rho(P), \dots, \rho^{4ks-1}(P)$ are $4ks$ edge disjoint copies of P_{2k+1} in H_1, where $\rho = (x_1 x_2 \dots x_{4ks})(y_1 y_2 \dots y_{4ks})(z_1 z_2 \dots z_{4ks})$ is a permutation on $V(H_1)$. Let H_0 be the subgraph of H_1 induced by those edges of H_1 which are not on these paths. Hence $E(H_0) = \{\bigcup_{i=2k}^{4ks-1} F_i(X, Y)\} \cup \{\bigcup_{j=2}^{4ks-1} F_j(X, Z)\}$, where $F_i(X, Y)$ (resp. $F_j(X, Z)$) denotes the 1-factor of jump i (resp. j) from X to Y (resp. from X to Z) in $K_{4ks,4ks}$. To complete the proof, it is enough to show that H_0 admits a C_{8ks}-decomposition. To obtain this cycle decomposition, consider $C'_i = F_{2k+2i}(X, Y) \oplus F_{2k+2i+1}(X, Y)$, $0 \leq i < \frac{4ks-2k}{2}$, and $C''_j = F_{2+2j}(X, Z) \oplus F_{2+2j+1}(X, Z)$, $0 \leq j < \frac{4ks-2}{2}$, where the addition in the subscript of F is taken modulo $4ks$. Clearly, C'_i and C''_j are cycles of length $8ks$ in H_0 and each of them can be partitioned into paths of length $2k$. This completes the proof. $\square$

Lemma 2.3. *The graph H_2 has a P_{2k+1}-decomposition for $k \geq 2$.*

Proof. Let H_2 be the graph defined above. Let $P = y_1 x_{4ks} y_2 x_{4ks-1} y_3 x_{4ks-2} \dots y_{k-1} x_{4ks-k+2} y_k x_{4ks-k+1} \infty$ be a path length $2k$ in H_2. Let $\rho = (\infty)(x_1 x_2 \dots x_{4ks})(y_1 y_2 \dots y_{4ks})$ be a permutation on $V(H_2)$. Then $P, \rho(P), \rho^2(P), \dots, \rho^{4ks-1}(P)$ are $4ks$ edge disjoint copies of P_{2k+1} in H_2. The set of edges of H_2 which are not on these paths is $\{\bigcup_{i=2k}^{4ks-1} F_i(X, Y)\} = H'$. As in the proof of the above lemma, to decompose the graph H' into paths of length $2k$, it is enough to find a C_{8ks}-decomposition of H'. Let $C'_i = F_{2k+2i}(X, Y) \oplus F_{2k+2i+1}(X, Y)$, $0 \leq i < \frac{4ks-2k}{2}$; $C'_0, C'_1, \dots, C'_{\frac{4ks-2k}{2}-1}$ decomposes H'; each C'_i is a cycle of length $8ks$ and it gives $4s$ edge disjoint paths of length $2k$. $\square$

3. P_{2k+1}-decomposition of $KG_{t,2}$

Now we are ready to prove Theorem 1.1.