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Note

Infinite end-devouring sets of rays with prescribed start vertices

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ABSTRACT

We prove that every end of a graph contains either uncountably many disjoint rays or a set of disjoint rays that meet all rays of the end and start at any prescribed feasible set of start vertices. This confirms a conjecture of Georgakopoulos.

1. Introduction

Looking for spanning structures in infinite graphs such as spanning trees or Hamilton cycles often involves difficulties that are not present when one considers finite graphs. It turned out that the concept of *ends* of an infinite graph is crucial for questions dealing with such structures. Especially for *locally finite graphs*, i.e., graphs in which every vertex has finite degree, ends allow us to define these objects in a more general topological setting [2].

Nevertheless, the definition of an end of a graph is purely combinatorial: We call one-way infinite paths *rays* and the vertex of degree 1 in them the *start vertex* of the ray. For any graph G we call two rays *equivalent* in G if they cannot be separated by finitely many vertices. It is easy to check that this defines an equivalence relation on the set of all rays in the graph G. The equivalence classes of this relation are called the *ends* of G and a ray contained in an end ω of G is referred to as an *ω-ray*.

When we focus on the structure of ends of an infinite graph G, we observe that *normal spanning trees* of G, i.e., rooted spanning trees of G such that the endvertices of every edge of G are comparable in the induced tree-order, have a powerful property: For any normal spanning tree T of G and every end ω of G there is a unique ω-ray in T which starts at the root of T and has the property that it meets every ω-ray of G, see [1, Sect. 8.2]. For any graph G, we say that an ω-ray with this property *devours* the end ω of G. Similarly, we say that a set of ω-rays *devours* ω if every ω-ray in G meets at least one ray out of the set. Note that if a set of ω-rays devours ω, then every ω-ray R meets the union of that set infinitely often, since each tail of R meets at least one ray out of the set.

End-devouring sets of rays are helpful for the construction of spanning structures such as infinite Hamilton cycles. For example, in a one-ended locally finite graph after removing an end-devouring set of rays, each component is finite. Thomassen [5] used this fact to show that the square of each locally finite 2-connected one-ended graph contains a spanning ray. Georgakopoulos [3] generalised this to locally finite 2-connected graphs with arbitrary many ends by building some other kind of spanning structure with the help of an end-devouring set of rays, which he then used to construct an infinite Hamilton cycle in the square of such a graph. He proved the following proposition about the existence of finite sets of rays devouring any *countable end*, i.e., an end which does not contain uncountably many disjoint rays. Note that the property of an end being countable is equivalent to the existence of a finite or countably infinite set of rays devouring the end.

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Proposition (*[3]*)**.** *Let G be a graph and ω be a countable end of G. If G has a set $\mathcal{R}$ of $k \in \mathbb{N}$ disjoint ω-rays, then it also has a set $\mathcal{R}'$ of k disjoint ω-rays that devours ω. Moreover, $\mathcal{R}'$ can be chosen so that its rays have the same start vertices as the rays in $\mathcal{R}$.*

For the proof of this proposition Georgakopoulos recursively applies a construction similar to the one yielding normal spanning trees to find rays for the end-devouring set. However, this proof strategy does not suffice to give a version of this proposition for infinitely many rays. He conjectured that such a version remains true [3, Problem 1]. We confirm this conjecture with the following theorem, which also covers the proposition above.

Theorem 1. *Let G be a graph, ω a countable end of G and $\mathcal{R}$ any set of disjoint ω-rays. Then there exists a set $\mathcal{R}'$ of disjoint ω-rays that devours ω and the start vertices of the rays in $\mathcal{R}$ and $\mathcal{R}'$ are the same.*

Note that, in contrast to the proposition, the difficulty of Theorem 1 for an infinite set $\mathcal{R}$ comes from fixing the set of start vertices, since any inclusion-maximal set of disjoint ω-rays devours ω.

After introducing some additionally needed terminology in Section 2, the proof of Theorem 1 will feature in Section 3. In Section 4 we will see why this theorem does not immediately extend to ends that contain an uncountable set of disjoint rays. There we discuss an additional necessary condition on the set of start vertices.

2. Preliminaries

All graphs in this paper are simple and undirected. For basic facts about finite and infinite graphs we refer the reader to [1]. If not stated differently, we also use the notation of [1].

We define the *union* $G \cup H$ of G and H as the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

Any ray T that is a subgraph of a ray R is called a *tail* of R. For a vertex v and an end ω of a graph G we say that a vertex set $X \subseteq V(G)$ *separates* v from ω if there does not exist any ω-ray that is disjoint from X and contains v.

For a finite set M of vertices of a graph G and an end ω of G, let $C(M, \omega)$ denote the unique component of $G - M$ that contains a tail of every ω-ray.

Given a path or ray Q containing two vertices v and w we denote the unique v – w path in Q by vQw. Furthermore, for Q being a v – w path we write $v\mathring{Q}$ for the path that is obtained from Q by deleting w.

For a ray R that contains a vertex v we write vR for the tail of R with start vertex v.

We use the following notion to abbreviate *concatenations* of paths and rays. Let P be a v – w path for two vertices v and w, and let Q be either a ray or another path such that $V(P) \cap V(Q) = \{w\}$. Then we write PQ for the path or ray $P \cup Q$, respectively. We omit writing brackets when stating concatenations of more than two paths or rays.

The *degree* of an end ω of G, denoted by $\deg(\omega)$, is the maximum cardinality of a set of disjoint ω-rays. Halin [4, Satz 1] showed that the degree of an end is well-defined. Note that an end is countable if and only if its degree is either finite or countably infinite.

3. Theorem

For the proof of Theorem 1 we shall use the following characterisation of ω-rays.

Lemma 2. *Let G be a graph, ω an end of G and $\mathcal{R}_{max}$ an inclusion-maximal set of pairwise disjoint ω-rays. A ray $R \subseteq G$ is an ω-ray, if and only if it meets rays of $\mathcal{R}_{max}$ infinitely often.*

Proof. Let W denote the set $\bigcup\{V(R)\,;\ R \in \mathcal{R}_{max}\}$.

If R is an ω-ray, then each tail of R meets a ray of $\mathcal{R}_{max}$ since $\mathcal{R}_{max}$ is inclusion-maximal. Hence R meets W infinitely often.

Suppose for a contradiction that R is an ω'-ray for an end $\omega' \neq \omega$ of G that contains infinitely many vertices of W. Let M be a finite set of vertices such that the two components $C := C(M, \omega)$ and $C' := C(M, \omega')$ of $G - M$ are different. By the pigeonhole principle there is either one ω-ray of $\mathcal{R}_{max}$ containing infinitely many vertices of $V(C') \cap V(R) \cap W$, or infinitely many disjoint rays of $\mathcal{R}_{max}$ containing those vertices. In both cases we get an ω-ray with a tail in C', since we cannot leave C' infinitely often through the finite set M. But this contradicts the definition of C. □

A natural strategy for constructing up to infinitely many disjoint rays is to inductively construct them in countably many steps. In each step we fix only finitely many finite paths as initial segments instead of whole rays, while extending previously fixed initial segments and ensuring that they can be extended to rays. This strategy is for example used by Halin [4, Satz 1] to prove that the maximum number of disjoint rays in an end is well-defined. Our proof of Theorem 1 is also based on that strategy. In order to guarantee that the set of rays we construct turns out to devour the end, we also fix an inclusion-maximal set of vertex disjoint rays of our specific end, so a countable set, and an enumeration of the vertices on these rays. Then we try in each step to either contain or separate the least vertex with respect to the enumeration that is not already dealt with from the end with appropriately chosen initial segments if possible. Otherwise, we extend a finite number of initial segments while still ensuring that all initial segments can be extended to rays. Although it is impossible to always contain or separate the considered vertex with our initial segments while being able to continue with the construction, it will turn out that the rays we obtain as the union of all initial segments actually do this.